ON WEIGHTED STIRLING AND OTHER RELATED NUMBERS AND SOME COMBINATORIAL APPLICATIONS

Ch. A. CHARALAMBIDES
University of Athens, Panepistemiopolis, Athens 621, Greece
(Submitted December 1982)

1. INTRODUCTION

The signless (absolute) Stirling numbers of the first kind

$$S_1(m, n) = (-1)^{m-n}s(m, n)$$

and the Stirling numbers of the second kind

$$S(m, n)$$

may be defined by

$$S_1(m, n) = (-1)^{m-n}\frac{1}{n!}[D^n(x)_m]_{x=0}, \quad S(m, n) = \frac{1}{n!}[\Delta^n x^m]_{x=0},$$

where $(x)_m = x(x-1)\ldots(x-m+1)$ denotes the falling factorial of degree m, D the differential operator, and Δ the difference operator. The numbers

$$C(m, n, r) = \frac{1}{n!}[\Delta^n(rx)_m]_{x=0}, \; r \text{ a real number},$$

which first arose as coefficients in the n-fold convolution of zero-truncated binomial (with r a positive integer) and negative binomial (with r a negative integer) distributions (see [1]) and have subsequently been studied systematically by the present author in [6], [7], and [8], are closely related to the Stirling numbers. This was the reason why Carlitz in [2] called the numbers

$$S_1(m, n|\lambda) = (-1)^{m-n}\lambda^{-n}C(m, n, \lambda), \quad S(m, n|\lambda) = \lambda^m C(m, n, \lambda^{-1})$$

degenerate Stirling numbers of the first and second kind, respectively.

Recently, Carlitz introduced and studied in [3] and [4] weighted Stirling numbers $\overline{S}_1(m, n, \lambda)$ and $\overline{S}(m, n, \lambda)$ by considering suitable combinatorial interpretations of $S_1(m, n)$ and $S(m, n)$, respectively. Several properties of these numbers and the related numbers

$$R_1(m, n, \lambda) = \overline{S}_1(m, n+1, \lambda) + S_1(m, n),$$

and

$$R(m, n, \lambda) = S(m, n+1, \lambda) + S(m, n)$$

were obtained.

In the present paper, by considering suitable combinatorial interpretations of the number $C(m, n, r)$ when r is a positive or negative integer, we introduce the weighted C-number, $\overline{C}(m, n; r, s)$, with r an integer and s a real number. Certain properties of these numbers are obtained in §2.

The related numbers

$$G(m, n; r, s) = \overline{C}(m, n+1; r, s) + C(m, n, r)$$

are shown to be equal to

$$G(m, n; r, s) = \frac{1}{n!}[\Delta^n(rx+s)_m]_{x=0}.$$

These numbers have been systematically studied in [9]. A representation of

$$G(m,\ m-n;\ r,\ s)$$

as the sum of binomial coefficients is obtained and certain properties of

$$G_m(r,\ s) = \sum_{n=0}^{m} G(m,\ n;\ r,\ s)$$

are derived in §3.

Combinatorial applications of the numbers

$$R_1(m,\ n,\ \lambda),\quad R(m,\ n,\ \lambda),\ \text{and}\ G(m,\ n;\ r,\ s)$$

are discussed in §4.

2. THE NUMBERS $\bar{C}(m,\ n;\ r,\ s)$

The C-numbers

$$C(m,\ n,\ r) = \frac{1}{n!}[\Delta^n (rx)_m]_{x=0}$$

may be expressed in the form (see [7]):

$$C(m,\ n,\ r) = \frac{m!}{n!}\sum_{\pi(m,\,n)} C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r), \tag{2.1}$$

where

$$C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r) = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\ k_2!\ \ldots\ k_m!}\binom{r}{1}^{k_1}\binom{r}{2}^{k_2}\cdots\binom{r}{m}^{k_m} \tag{2.2}$$

and the summation is over all partitions $\pi(m,\ n)$ of m in n parts, that is, all nonnegative integer solutions $(k_1,\ k_2,\ \ldots,\ k_m)$ of the equations

$$k_1 + 2k_2 + \cdots + mk_m = m,\quad k_1 + k_2 + \cdots + k_m = n. \tag{2.3}$$

Note that $C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ r)$, r a positive integer, is a distribution of (number of ways of putting) m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of capacity limited to one ball, such that k_j cells contain exactly j balls each, $j = 1,\ 2,\ \ldots,\ m$. When the capacity of each cell is unlimited, the corresponding number is equal to

$$|C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ -r)| = (-1)^m C(m;\ k_1,\ k_2,\ \ldots,\ k_m;\ -r),$$

where r is a positive integer.

The expression (2.1) leads to the following combinatorial interpretations of the C-numbers:

$$\frac{m!}{n!}\ C(m,\ n,\ r),\ r \text{ a positive integer},$$

is the number of ways of putting m like balls into n different cells, each of which has r different compartments of capacity limited to one ball, with no cell empty. When the capacity of each compartment is unlimited, the corresponding number is equal to

$$\frac{m!}{n!}|C(m,\ n,\ -r)| = (-1)^m \frac{n!}{m!} C(m,\ n,\ -r),\ r \text{ a positive integer}.$$

Consider the weighted number of distributions

$$C(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, r,\, s) = \frac{m!}{(k_1 + k_2 + \cdots + k_m)!} \sum (k_1w_1 + k_2w_2 + \cdots + k_mw_m) \tag{2.4}$$

where the weights

$$w_j = w_j(r,\, s) = (s)_j/(r)_j,\ j = 1,\, 2,\, \ldots,\, m,\ r \text{ a positive integer, } s \text{ a real number,}$$

and the summation is over all distributions of m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of capacity limited to one ball, such that k_j cells contain exactly j balls each, $j = 1, 2, \ldots, m$, and

$$\overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, -r,\, s) = \frac{m!}{(k_1 + k_2 + \cdots + k_m)!} \sum (k_1v_1 + k_2v_2 + \cdots + k_mv_m) \tag{2.5}$$

where the weights

$$v_j = v_j(-r,\, s) = (s)_j/(-r)_j,\ j = 1,\, 2,\, \ldots,\, m,\ r \text{ a positive integer, } s \text{ a real number,}$$

and the summation is over all distributions of m like balls into $k_1 + k_2 + \cdots + k_m$ different cells, each of which has r different compartments of unlimited capacity, such that k_j cells contain exactly j balls each, $j = 1, 2, \ldots, m$.

Let

$$\overline{C}(m,\, n;\, r,\, s) = \sum_{\pi(m,\, n)} \overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k\ ;\, r,\, s),\ r \text{ an integer, } s \text{ a real number,} \tag{2.6}$$

where the summation is over all partitions $\pi(m, n)$ of m in n parts. The numbers

$$C(m,\, n;\, r,\, s) = \frac{1}{n}\, \overline{C}(m,\, n;\, r,\, s) \tag{2.7}$$

may be called *weighted C-numbers*.

Putting $s = r$ in (2.4) and (2.6), with $w_j = 1$, $j = 1, 2, \ldots, m$, we obtain

$$C(m,\, n;\, r,\, r) = C(m,\, n,\, r), \tag{2.8}$$

while putting $s = -r$ in (2.5) and (2.6), with $v_j = 1$, $j = 1, 2, \ldots, m$, we get

$$(-1)^m C(m,\, n;\, -r,\, -r) = (-1)^m C(m,\, n,\, -r) = |C(m,\, n,\, -r)|. \tag{2.9}$$

Now consider the generating function

$$\overline{F}(t,\, u_1,\, u_2,\, \ldots;\, r,\, s) = \sum_{m=0}^{\infty} \sum_{\pi(m)} \overline{C}(m;\, k_1,\, k_2,\, \ldots,\, k_m;\, r,\, s)\, \frac{t^m}{m!} u_1^{k_1} u_2^{k_2} \ldots\, u_m^{k_m},\ r \text{ an integer, } s \text{ a real number,}$$

where the inner summation is over all partitions $\pi(m)$ of m, that is, over all nonnegative integer solutions $(k_1, k_2, \ldots, k_m)$ of the equation

$$k_1 + 2k_2 + \cdots + mk_m = m.$$

Using (2.4) when r is a positive integer and (2.5) when r is a negative integer, we get

$$\bar{F}(t, u_1, u_2, \ldots; r, s)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\pi(m)} (k_1 w_1 + k_2 w_2 + \cdots + k_m w_m) \frac{m!}{k_1! k_2! \ldots k_m!} \left[\binom{r}{1} u_1 t\right]^{k_1} \left[\binom{r}{2} u_2 t\right]^{k_2} \cdots \left[\binom{r}{m} u_m t\right]^{k_m}$$

$$= \left\{\binom{s}{1} u_1 t + \binom{s}{2} u_2 t^2 + \cdots + \binom{s}{m} u_m t^m + \cdots\right\} \exp\left\{\binom{r}{1} u_1 t + \binom{r}{2} u_2 t^2 + \cdots + \binom{r}{m} u_m t^m + \cdots\right\}.$$

The generating function

$$\begin{aligned} \bar{F}(t, u; r, s) &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \bar{C}(m, n; r, s) \frac{t^m}{m!} u^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \bar{C}(m, n; r, s) \frac{t^m}{m!} u^n \end{aligned} \tag{2.10}$$

may be obtained from $\bar{F}(t, u_1, u_2, \ldots; r, s)$ by putting $u_j = u$, $j = 1, 2, \ldots$. We get

$$\bar{F}(t, u; r, s) = u[(1 + t)^s - 1]\exp\{u[(1 + t)^r - 1]\}, \tag{2.11}$$

and

$$\bar{f}\,(t; r, s) = \sum_{m=n}^{\infty} \bar{C}(m, n; r, s) \frac{t^m}{m!} = \frac{1}{(n-1)!}[(1+t)^s - 1][(1+t)^r - 1]^{n-1}. \tag{2.12}$$

The corresponding generating function of the usual C-numbers is ([7]):

$$f_n(t; r) = \sum_{m=n}^{\infty} C(m, n, r) \frac{t^m}{m!} = \frac{1}{n!}[(1 + t)^r - 1]^n. \tag{2.13}$$

Since

$$\bar{f}_n(t; r, s) = [(1 + t)^s - 1] f_{n-1}(t; r),$$

we find

$$\bar{C}(m, n; r, s) = \sum_{j=1}^{m-n+1} \binom{m}{j} (s)_j\, C(m - j, n - 1, r). \tag{2.14}$$

Note that (2.12) for $s = r$ reduces to

$$\bar{f}_n(t; r, s) = n f_n(t; r),$$

which implies (2.8) and (2.9).

Using the relation ([7]),

$$(s)_j = \sum_{k=1}^{j} C(j, k, r)(s/r)_k.$$

(2.14) may be written as

$$\begin{aligned} \bar{C}(m, n; r, s) &= \sum_{j=1}^{m-n+1} \binom{m}{j} \left\{\sum_{k=1}^{j} C(j, k, r)(s/r)_k\right\} C(m - j, n - 1, r) \\ &= \sum_{k=1}^{m-n-1} \left\{\sum_{j=k}^{m} \binom{m}{j} C(j, k, r) C(m - j, n - 1, r)\right\} (s/r)_k. \end{aligned}$$

From (2.13), we have

$$\binom{k+n}{k} f_{k+n}(t;\ r) = f_k(t;\ r) f_n(t;\ r),$$

which implies

$$\binom{k+n}{k} C(m,\ k+n,\ r) = \sum_{j=k}^{m} \binom{m}{j} C(j,\ k,\ r) C(m-j,\ n,\ r).$$

Therefore,

$$\bar{C}(m,\ n;\ r,\ s) = \sum_{k=1}^{m-n-1} \binom{n+k-1}{k} C(m,\ n+k-1,\ r)(s/r)_k. \tag{2.15}$$

Using the generating functions (see [3]),

$$\bar{g}_n(t,\ \lambda) = \sum_{m=n}^{\infty} \bar{S}_1(m,\ n,\ \lambda)\frac{t^m}{m!} = \frac{1}{(n-1)!}[(1-t)^{-\lambda} - 1][-\log(1-t)]^{n-1}, \tag{2.16}$$

and

$$h_n(t) = \sum_{m=n}^{\infty} S(m,\ n)\frac{t^m}{m!} = \frac{1}{n!}(e^t - 1)^n.$$

(2.12) may be expressed as

$$\begin{aligned}
\bar{f}_n(t;\ r,\ s) &= \sum_{m=n}^{\infty} \bar{C}(m,\ n;\ r,\ s)\frac{t^m}{m!} = \frac{1}{(n-1)!}[(1+t)^s - 1][e^{r\log(1+t)} - 1]^{n-1} \\
&= \sum_{k=n}^{\infty} S(k-1,\ n-1) r^{k-1}\left\{\frac{1}{(k-1)!}[(1+t)^s - 1][\log(1+t)]^{k-1}\right\} \\
&= \sum_{k=n}^{\infty} r^{k-1} S(k-1,\ n-1) \sum_{m=n}^{\infty} (-1)^{m-k-1}\bar{S}_1(m,\ k,\ -s)\frac{t^m}{m!} \\
&= \sum_{m=n}^{\infty} \left\{\sum_{k=n}^{m} (-1)^{m-k-1} r^{k-1}\bar{S}_1(m,\ k,\ -s) S(k-1,\ n-1)\right\}\frac{t^m}{m!};
\end{aligned}$$

hence,

$$\bar{C}(m,\ n;\ r,\ s) = \sum_{k=n}^{m} (-1)^{m-k+1} r^{k-1}\bar{S}_1(m,\ k,\ -s) S(k-1,\ n-1). \tag{2.17}$$

Again from (2.12) we have

$$\lim_{r\to 0} r^{-n+1}\bar{f}_n(t;\ r,\ s) = \frac{1}{(n-1)!}[(1+t)^s - 1][\log(1+t)]^{n-1}$$

and

$$\lim_{r\to\infty} \bar{f}_n(t/r;\ r,\ s) = \frac{1}{(n-1)!}(e^{\lambda t} - 1)(e^t - 1)^{n-1}, \text{ if } \lim_{r\to\infty}\frac{s}{r} = \lambda,$$

which, by virtue of the generating functions of the weighted Stirling numbers, (2.16), and (see [3])

$$\bar{h}(t,\ \lambda) = \sum_{m=n}^{\infty} S(m,\ n,\ \lambda)\frac{t^m}{m!} = \frac{1}{(n-1)!}(e^{\lambda t} - 1)(e^t - 1)^{n-1}, \tag{2.18}$$

imply

$$\lim_{r\to 0} r^{-n+1}\bar{C}(m,\ n;\ r,\ s) = (-1)^{m-n+1} S_1(m,\ n,\ -s) \tag{2.19}$$

and

$$\lim_{r\to\infty} r^{-m}\bar{C}(m,\ n;\ r,\ s) = S(m,\ n,\ \lambda), \text{ if } \lim_{r\to\infty}\frac{s}{r} = \lambda, \tag{2.20}$$

respectively.

3. THE NUMBERS $G(m, n; r, s)$

Let

$$G(m, n; r, s) = \bar{C}(m, n+1; r, s) + C(m, n, r). \tag{3.1}$$

Then (2.14) implies

$$G(m, n; r, s) = \sum_{j=0}^{m-n} \binom{m}{j} (s)_j C(m-j, n, r). \tag{3.2}$$

Since

$$C(m, n, r) = \frac{1}{n!}[\Delta^n (rx)_m]_{x=0}, \; n = 0, 1, 2, \ldots, m, \; m = 0, 1, 2, \ldots$$

and

$$C(m, n, r) = 0 \text{ for } m < n,$$

it follows that

$$G(m, n; r, s) = \sum_{j=0}^{m} \binom{m}{j} (s)_j C(m-j, n, r) = \frac{1}{n!}\Delta^n \left[\sum_{j=0}^{m} \binom{m}{j} (s)_j (rx)_{m-j}\right]_{x=0}$$

and, by virtue of Vandermonde's convolution formula,

$$G(m, n; r, s) = \frac{1}{n!}[\Delta^n (rx + s)_m]_{x=0} = \frac{1}{n!}\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (rk + s)_m.$$

These numbers, shown as coefficients in a generalization of the Hermite polynomials considered by Gould and Hopper, were systematically studied in [9]. A representation of $G(m, m-n; r, s)$ as the sum of binomial coefficients will be discussed here.

The numbers $G(m, n; r, s)$ satisfy the triangular recurrence relation

$$G(m+1, n; r, s) = (rn + s - m)G(m, n; r, s) + rG(m, n-1, r) \tag{3.3}$$

with initial conditions

$$G(0, n; r, s) = \delta_{0n}, \; G(m, 0; r, s) = (s)_m, \text{ and } G(m, n; r, s) = 0 \text{ for } m < n.$$

Putting $n = m + 1$, we get

$$G(m+1, m+1; r, s) = rG(m, m; r, s), \; m = 0, 1, 2, \ldots$$

and

$$G(m, m; r, s) = r^m. \tag{3.4}$$

If we put $n = 1$ in (3.3), we find

$$G(m+1, 1; r, s) = (r + s - m)G(m, 1; r, s) + r(s)_m$$

and, in particular,

$$G(2, 1; r, s) = (r + s - 1)r + rs = r(r + 2s - 1).$$

Again, if we put $n = m - k + 1$ in (3.3), we obtain

$$\begin{aligned} &G(m+1, m+1-k; r, s) - rG(m, m-k; r, s) \\ &= [r(m-k+1) + s - m]G(m, m-k+1; r, s) \end{aligned}$$

or

$$\begin{aligned} &\Delta_m r^{-m+k} G(m, m-k; r, s) \\ &= r^{-m+k-1}[(r-1)m - r(k-1) + s]G(m, m-k+1; r, s). \end{aligned} \tag{3.5}$$

For $k = 1$, we have

$$\Delta_m r^{-m+1} G(m, m-1; r, s) = (r-1)m + s$$

and

$$r^{-m+1}G(m,\ m-1;\ r,\ s) = \Delta_m^{-1}[(r-1)m+s] = (r-1)\binom{m}{2} + s\binom{m}{1} + K.$$

Since $G(2, 1; r, s) = r(r + 2s - 1)$, $K = 0$, and

$$r^{-m+1}G(m,\ m-1;\ r,\ s) = (r-1)\binom{m}{2} + s\binom{m}{1}. \tag{3.6}$$

Taking $k = 2$ in (3.5), we get

$$r^{-m+2}G(m,\ m-2;\ r,\ s) = \Delta_m^{-1}\left\{[(r-1)m+s-r]\left[(r-1)\binom{m}{2} + s\binom{m}{1}\right]\right\},$$

which on using the relations

$$\Delta^{-1}\binom{m}{j} = \binom{m}{j+1},$$

$$\Delta_m^{-1}\left\{m\binom{m}{j}\right\} = m\binom{m}{j+1} - \binom{m+1}{j+2} = (j+1)\binom{m}{j+2} + j\binom{m}{j+1},$$

gives

$$r^{-m+2}G(m,\ m-2;\ r,\ s) = 3(r-1)^2\binom{m}{4} + (r-1)(r+3s-2)\binom{m}{3} + s(s-1)\binom{m}{2}.$$

Hence, $r^{-m+2}G(m, m-2; r, s)$ is a polynomial of m of degree 4. Consequently, $r^{-m+n}G(m, n-n; r, s)$ will be a polynomial of m of degree $2n$. Let us write it as follows:

$$r^{-m+n}G(m,\ m-n;\ r,\ s) = \sum_{k=0}^{2n} H(n,\ k;\ r,\ s)\binom{m}{2n-k}.$$

Multiplying both numbers by $[(r-1)m - rn + s]$ and using (3.5), we have

$$\Delta_m r^{-m+n+1}G(m,\ m-n-1;\ r,\ s) = \sum_{k=0}^{2n} H(n,\ k;\ r,\ s)[(r-1)m - rn + s]\binom{m}{2n-k},$$

and since

$$\Delta_m^{-1}[(r-1)m - rn + s]\binom{m}{2n-k}$$

$$= (r-1)(2n-k+1)\binom{m}{2n-k+2} + [(r-1)(n-k) - n + s]\binom{m}{2n-k+1},$$

we get

$$r^{-m+n+1}G(m,\ m-n-1;\ r,\ s)$$

$$= \sum_{k=0}^{2n}(2n-k+1)(r-1)H(n,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$+ \sum_{k=0}^{2n}[(r-1)(n-k) - n + s]H(n,\ k;\ r,\ s)\binom{m}{2n-k+1} + K$$

and

$$\sum_{k=0}^{2n+2} H(n+1,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$= \sum_{k=0}^{2n}(2n-k+1)(r-1)H(n,\ k;\ r,\ s)\binom{m}{2n-k+2}$$

$$+ \sum_{k=1}^{2n+1}[(n-k+1)(r-1) - n + s]H(n,\ k-1;\ r,\ s)\binom{m}{2n-k+2} + K.$$

Therefore,

$$H(n+1, k; r, s) = (2n-k+1)(r-1)H(n, k; r, s) + [(n-k+1)(r-1) - n + s]H(n, k-1; r, s) \tag{3.7}$$

and

$$H(n+1, 2n+2; r, s) = K.$$

From (3.6), it follows that

$$H(1, 0; r, s) = r - 1,\ H(1, 1; r, s) = s, \text{ and } H(1, k; r, s) = 0 \text{ for } k > 1.$$

Putting successively $n = 1, 2, \ldots$ in (3.7), we conclude that

$$H(n, k; r, s) = 0 \text{ if } k > n,$$

and hence,

$$r^{-m+n}G(m, m-n; r, s) = \sum_{k=0}^{n} H(n, k; r, s)\binom{m}{2n-k}. \tag{3.8}$$

Using (3.7), we may easily deduce that

$$H(n, n; r, s) = (s)_n,\ n = 1, 2, \ldots, \tag{3.9}$$

and

$$H(n, 0; r, s) = (r-1)^n 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) = (r-1)^n \frac{(2n)!}{n! 2^n} \tag{3.10}$$

Moreover, for

$$S_n(r, s) = \sum_{k=0}^{n} (-1)^{n-k} H(n, k; r, s)$$

we get

$$S_n(r, s) = [(s - r + 1) - r(n-1)]S_{n-1}(r, s),\ n = 2, 3, \ldots,$$

and

$$S_1(r, s) = -H(1, 0; r, s) + H(1, 1; r, s) = s - r + 1.$$

Therefore,

$$S_n(r, s) = \sum_{k=0}^{n} (-1)^{n-k} H(n, k; r, s) = r^n\left(\frac{s-r+1}{r}\right)_n. \tag{3.11}$$

Multiplying both members of (3.8) by $(-1)^{m+j}\binom{2n-j}{m}$ and summing for $m = n, n+1, \ldots, 2n-j$, we obtain the relation

$$H(n, j; r, s) = \sum_{m=n}^{2n-j} (-1)^{m+j}\binom{2n-j}{m} r^{-m+n}G(m, m-n; r, s), \tag{3.12}$$

which leads to interesting combinatorial interpretations for these numbers or, more precisely, for the numbers

$$G_2(m, n; r, s) = r^n H(m-n, m-2n; r, s) = \sum_{k=0}^{n} (-1)^k \binom{m}{k} r^k G(m-k, n-k; r, s). \tag{3.13}$$

Since (see [9])

$$\sum_{m=n}^{\infty} G(m, n; r, s)\frac{t^m}{m!} = \frac{1}{n!}(1+t)^s[(1+t)^r - 1]^n$$

it follows that

$$\sum_{m=n}^{\infty} G_2(m, n; r, s)\frac{t^m}{m!} = \sum_{m=n}^{\infty}\left\{\sum_{k=0}^{n} (-1)^k\binom{m}{k} r^k G(m-k, n-k; r, s)\right\}\frac{t^m}{m!}$$

(continued)

$$= \sum_{k=0}^{n} (-1)^k \frac{(rt)^k}{k!} \sum_{m=n}^{\infty} G(m-k;\ n-k,\ r,\ s)\frac{t^{m-k}}{(m-k)!}$$

$$= \frac{1}{n!}(1+t)^s \sum_{k=0}^{n} \binom{n}{k} [(1+t)^r - 1]^{n-k}(-rt)^k,$$

and

$$\sum_{m=n}^{\infty} G_2(m,\ n;\ r,\ s)\frac{t^m}{m!} = \frac{1}{n!}(1+t)^s[(1+t)^r - 1 - rt]^n. \tag{3.14}$$

Consider n different cells of r different compartments each and a (control) cell of s different compartments. The compartments may be of limited capacity or not (Riorday [11, Ch. 5]). From (3.14), it follows that the number of ways of putting m like balls into these cells such that each cell among the first n contains at least two balls is equal to

$$\frac{n!}{m!} G_2(m,\ n;\ r,\ s)$$

when the capacity of each compartment is limited to one ball and to

$$(-1)^m \frac{n!}{m!} G_2(m,\ n;\ -r,\ -s)$$

when the capacity of each compartment is unlimited.

It is worth noting that the expression (3.8) may be written in the form

$$r^{-m+n} G(m,\ m-n;\ r,\ s) = \sum_{j=0}^{n} L(n,\ j;\ r,\ s)\binom{m+j}{2n}, \tag{3.15}$$

where, on using the relation

$$\binom{m+j}{2n} = \sum_{k=0}^{j} \binom{j}{k}\binom{m}{2n-k}$$

the coefficients $L(n,\ j;\ r,\ s)$ are related to the coefficients $H(n,\ k;\ r,\ s)$ by

$$H(n,\ k;\ r,\ s) = \sum_{j=k}^{n} \binom{j}{k} L(n,\ j;\ r,\ s), \tag{3.16}$$

$$L(n,\ j;\ r,\ s) = \sum_{k=j}^{n} (-1)^{k-j}\binom{k}{j} H(n,\ k;\ r,\ s). \tag{3.17}$$

Moreover, $L(n,\ j;\ r,\ s)$ satisfy the recurrence relation

$$\begin{aligned} L(n+1,\ j;\ r,\ s) = {} & [(r-1)(n+j+1) + n - s]L(n,\ j;\ r,\ s) \\ & + [(r-1)(n-j+1) - n + s]L(n,\ j-1;\ r,\ s), \end{aligned} \tag{3.18}$$

with initial conditions

$L(1,\ 0;\ r,\ s) = r - s - 1$, $L(1,\ 1;\ r,\ s) = s$, and $L(n,\ j;\ r,\ s) = 0$ if $j > n$.

Also, by (3.9), (3.10), and (3.11),

$$L(n,\ n;\ r,\ s) = H(n,\ n;\ r,\ s) = (s)_n,\ n = 1,\ 2,\ \ldots, \tag{3.19}$$

$$L(n,\ 0;\ r,\ s) = \sum_{k=0}^{n} (-1)^k H(n,\ k;\ r,\ s) = (-1)^n r^n \left(\frac{s-r+1}{r}\right)_n, \tag{3.20}$$

$$\sum_{j=0}^{n} L(n,\ j;\ r,\ s) = H(n,\ 0;\ r,\ s) = (r-1)^n \frac{(2n)!}{n!\,2^n} \tag{3.21}$$

We conclude this section by considering the sum

$$G_m(r,\ s) = \sum_{n=0}^{m} G(m,\ n;\ r,\ s), \tag{3.22}$$

which for $s = 0$ reduces to

$$C_m(r) = \sum_{n=0}^{m} C(m,\ n,\ r). \tag{3.23}$$

This sum has been studied in [5] and also by Carlitz in [2] as

$$A_m(\lambda) = \sum_{n=0}^{m} S(m,\ n|\lambda) = \sum_{n=0}^{m} \lambda^m C(m,\ n,\ 1/\lambda) = \lambda^m C_m(1/\lambda).$$

Note that, since (see [7])

$$\lim_{r\to\infty} r^{-m} C(m,\ n,\ r) = S(m,\ n), \tag{3.24}$$

it follows that

$$\lim_{r\to\infty} r^{-m} C_m(r) = \sum_{n=0}^{m} S(m,\ n) = B_m, \tag{3.25}$$

where B_m denotes the Bell number. Also from (3.1) we get, on using (2.20) and (3.24),

$$\lim_{r\to\infty} r^{-m} G(m,\ n;\ r,\ s) = \overline{S}(m,\ n+1,\ \lambda) + S(m,\ n)$$

$$= R(m,\ n,\ \lambda),\ \lambda = \lim_{r\to\infty} \frac{s}{r}.$$

Hence,

$$\lim_{r\to\infty} r^{-m} G_m(r,\ s) = \sum_{n=0}^{m} R(m,\ n,\ \lambda) = B_m(\lambda),\ \lambda = \lim_{r\to\infty} \frac{s}{r}, \tag{3.26}$$

where the number $B_m(\lambda)$ has been discussed by Carlitz in [3].

Now, from (3.22), (3.23), and (3.2), it follows that

$$G_m(r,\ s) = \sum_{n=0}^{m} \sum_{j=0}^{m-n} \binom{m}{j} (s)_j\, C(m-j,\ n,\ r) = \sum_{j=0}^{m} \binom{m}{j} (s)_j \sum_{n=0}^{m-j} C(m-j,\ n,\ r),$$

$$G_m(r,\ s) = \sum_{j=0}^{m} \binom{m}{j} (s)_j\, C_{m-j}(r), \tag{3.27}$$

and

$$F(t;\ r,\ s) = \sum_{m=0}^{\infty} G_m(r,\ s)\frac{t^m}{m!} = \sum_{j=0}^{s} \binom{s}{j} t^j \sum_{m=0}^{\infty} C_m(r)\frac{t^m}{m!}$$

$$= (1+t)^s \exp\{(1+t)^r - 1\}, \tag{3.28}$$

since (see [5] or [2])

$$F(t;\ r) = \sum_{m=0}^{\infty} C_m(r)\frac{t}{m!} = \exp\{(1+t)^r - 1\}.$$

We have

$$F(t;\ r,\ s+1) = (1+t)F(t;\ r,\ s)$$

and, hence,

$$G_m(r,\ s+1) = G_m(r,\ s) + mG_{m-1}(r,\ s),\ m = 1,\ 2,\ \ldots,\ G_0(r,\ s) = 1. \tag{3.29}$$

Differentiation of (3.29) gives the differential equation

$$(1 + t)\frac{d}{dt} F(t;\ r,\ s) = sF(t;\ r,\ s) + r(1 + t)^r F(t;\ r,\ s),$$

which implies

$$G_{m+1}(r,\ s) = (s - m)G_m(r,\ s) + r\sum_{j=0}^{m}\binom{m}{j}(r)_j G_{m-j}(r,\ s). \tag{3.30}$$

Writing the generating function $F(t;\ r,\ s)$ in the form

$$F(t;\ r,\ s) = e^{-1}(1 + t)^s \exp\{(1 + t)^r\} = e^{-1}\sum_{k=0}^{\infty}\frac{(1 + t)^{rk+s}}{k!}$$

$$= e^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m=0}^{\infty}(rk + s)_m\frac{t^m}{m!},$$

we find

$$G_m(r,\ s) = e^{-1}\sum_{k=0}^{\infty}\frac{(rk + s)_m}{k!} \tag{3.31}$$

which should be compared to Dobinski's formula for the Bell number:

$$B_m = e^{-1}\sum_{k=0}^{\infty}\frac{k^m}{k!}. \tag{3.32}$$

From (3.31) we obtain, on using (3.32) and the relation (see Carlitz [3]),

$$(rk + s)_m = \sum_{n=0}^{m}(-1)^{m-n}R_1(m,\ n,\ -s)r^n k^n,$$

$$G_m(r,\ s) = \sum_{n=0}^{m}(-1)^{m-n}R_1(m,\ n,\ -s)r^n B_n. \tag{3.33}$$

4. COMBINATORIAL APPLICATIONS

4.1 Modified Occupancy Stirling Distributions of the First Kind

Consider an urn containing r identical balls from each of $n + v$ different kinds (colors). Suppose that m balls are drawn one after the other and after each drawing the chosen ball is returned togather with another ball of the same kind (color). Let X be the number of kinds (colors) among n specified appearing in the sample. The probability function of X, on using the sieve (inclusion-exclusion) formula, may be obtained as

$$p_1(k;\ m,\ n,\ r,\ v) = Pr(X + k)$$

$$= \binom{n}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{rj + rv + m - 1}{m}\Big/\binom{rn + rv + m - 1}{m}$$

$$= \frac{(n)_k}{(rn + rv + m - 1)_m}|G(m,\ k;\ -r,\ -rv)|, \tag{4.1}$$

$$k = 1,\ 2,\ \ldots,\ \min\{m,\ n\}.$$

Now, consider the case where the number m of balls is not fixed but balls are sequentially drawn and after each drawing the chosen ball is returned together with another ball of the same kind until a predetermined number k of kinds among the n specified is represented in the sample. Let Y be the number of balls required. Then the probability function of Y is given by

$$q_1(m;\ k,\ n,\ r,\ v) = p_1(k-1;\ m-1,\ n,\ r,\ v)\frac{r(n-k+1)}{rn+rv+m-1}$$

$$= \frac{(n)_{k-1}}{(rn+rv+m-2)_{m-1}}|G(m-1,\ k-1;\ -r,\ -rv)|\frac{r(n-k+1)}{rn+rv+m-1}$$

$$= \frac{r(n)_k}{(rn+rv+m-1)_m}|G(m-1,\ k-1;\ -r,\ -rv)|, \tag{4.2}$$

$$m = k,\ k+1,\ \ldots .$$

Suppose that $\lim_{r\to 0} rn = \theta$ and $\lim_{r\to 0} rv = \lambda$, then since (see [9])

$$\lim_{r\to 0} r^{-k}|G(m,\ k;\ -r,\ -rv)| = |s(m,\ k,\ \lambda)| = S_1(m,\ k,\ \lambda)$$

it follows from (4.1) and (4.2) that

$$p_1(k;\ m,\ \theta,\ \lambda) = \lim_{r\to 0} p_1(k;\ m,\ n,\ r,\ v) = \frac{(\theta)_k}{(\theta+\lambda+m-1)_m} S_1(m,\ k,\ \lambda), \tag{4.3}$$

and

$$q_1(m;\ k,\ \theta,\ \lambda) = \lim_{r\to 0} q_1(m;\ k,\ n,\ r,\ v)$$

$$= \frac{(\theta)_k}{(\theta+\lambda+m-1)_m} S_1(m-1,\ k-1,\ \lambda). \tag{4.4}$$

Note that (4.3) gives in particular $\lambda = 0$ the occupancy Stirling distribution of the first kind (cf. Johson and Kotz [10, p. 246]).

4.2 Modified Occupancy Stirling Distributions of the Second Kind

Suppose that m distinct balls are randomly allocated into $n + r$ different cells and let X be the number of occupied cells (by at least one ball) among n specified. Since $R(m,\ k,\ r)$ is the number of ways of putting the m balls into the $n + r$ cells such that k cells among the n specified are occupied (by at least one ball), it follows that

$$Pr(X = k) = \frac{(n)_k}{(n+r)^m} R(m,\ k,\ r),\ k = 1,\ 2,\ \ldots,\ \min\{m,\ n\}. \tag{4.5}$$

The factorial moments of X may be obtained in terms of the number $R(m, k, r)$ as follows:

$$\mu_{(j)} = \sum_{k=j}^{n}(k)_j Pr(X = k) = \frac{1}{(n+r)^m}\sum_{k=r}^{n}(k)_j (n)_k R(m,\ k,\ r)$$

$$= \frac{\binom{n}{j}}{(n+r)^m}\sum_{k=j}^{n}\binom{n-j}{k-j}\frac{k!}{j!} R(m,\ k,\ r)$$

(continued)

$$= \frac{\binom{n}{j}}{(n+r)^m} \sum_{i=0}^{n-j} \binom{n-j}{i} (i+j)_i R(m,\ i+j,\ r).$$

Since

$$\sum_{i=0}^{n-j} \binom{n-j}{i} (i+j)_i R(m,\ i+j,\ r) = \frac{1}{j!} \sum_{i=0}^{n-j} \binom{n-j}{i} \Delta^{i+j} r^m = \frac{1}{j!} \Delta^j E^{n-j} r^m$$

$$= \frac{1}{j!} \Delta^j (r+n-j)^m = R(m,\ j,\ r+n-j),$$

$$\mu_{(j)} = \frac{1}{(n+r)^m} \binom{n}{j} R(m,\ j,\ r+n-j). \tag{4.6}$$

Now, consider the case where the number of balls is not fixed but balls are sequentially (one after the other) allocated into the $n + r$ different cells until a predetermined number k of cells among the n specified are occupied. Let Y be the number of balls required. Then,

$$Pr(Y = m) = \frac{(n)_{k-1}}{(n+r)^{m-1}} R(m-1,\ k-1,\ r) \frac{n-k+1}{n+r}$$

$$= \frac{(n)_k}{(n+r)^m} R(m-1,\ k-1,\ r),\ m = k,\ k+1,\ \ldots .$$

Since $\sum_{m=k}^{\infty} Pr(Y = m) = 1$, we must have

$$\sum_{m=k}^{\infty} R(m-1,\ k-1,\ r) \frac{1}{(n+r)^m} = \frac{1}{(n)_k}.$$

This relation holds in the more general case where r is any real number and n real number different from 0, 1, 2, ..., $k - 1$. Indeed from Carlitz [3],

$$\sum_{m=k}^{\infty} R(m,\ k,\ \lambda) z^m = \frac{z^k}{(1-\lambda z)(1-(\lambda+1)z) \ldots (1-(\lambda+k)z)}$$

it follows that

$$\sum_{m=k}^{\infty} R(m-1,\ k-1,\ r) z^{m-1} = \frac{1}{(z^{-1}-\lambda)(z^{-1}-\lambda-1) \ldots (z^{-1}-\lambda-k+1)} \quad \frac{1}{(z^{-1}-\lambda)_k}$$

and putting $z^{-1} - \lambda = n$, $z = (n + \lambda)^{-1}$, we obtain

$$\sum_{m=k}^{\infty} R(m-1,\ k-1,\ \lambda) \frac{1}{(m+\lambda)^m} = \frac{1}{(n)_k}.$$

Remark 4.1

The distribution (4.5) with r not necessarily a positive integer arose in the following randomized occupancy problem (see [10, p. 139]). Suppose that m balls are randomly allocated into n different cells and that each ball has probability p of staying in its cell and probability $q = 1 - p$ of leaking. Let X be the number of occupied cells. Then, the probability function of X may be obtained by using the sieve (inclusion-exclusion) formula in the form

$$Pr(X = k) = \binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (q + p(k - j)/n)^m$$

$$= \frac{(n)_k}{(n + \lambda)^m} R(m, k, \lambda),\ k = 1, 2, \ldots, \min\{m, n\},\ \lambda = nq/p.$$

REFERENCES

1. T. Cacoullos & Ch. A. Charalambides. "On Minimum Variance Unbiased Estimation for Truncated Binomial and Negative Binomial Distributions." *Ann. Inst. Statist. Math.* 27 (1975):235-44.
2. L. Carlitz. "Degenerate Stirling, Bernoulli and Eulerian Numbers." *Utilitas Mathematica* 15 (1979):51-88.
3. L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind--I." *The Fibonacci Quarterly* 18, no. 2 (1980):147-62.
4. L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind--II." *The Fibonacci Quarterly* 18, no. 3 (1980):242-57.
5. Ch. A. Charalambides. "A Sequence of Exponential Integers." *Bull. Soc. Math. Grèce* 15 (1974):52-58.
6. Ch. A. Charalambides. "The Asymptotic Normality of Certain Combinatorial Distributions." *Ann. Inst. Statist. Math.* 28 (1976):499-506.
7. Ch. A. Charalambides. "A New Kind of Numbers Appearing in the n-Fold Convolution of Truncated Binomial and Negative Binomial Distributions." *SIAM J. Appl. Math.* 33 (1977):279-88.
8. Ch. A. Charalambides. "Some Properties and Applications of the Differences of the Generalized Factorials." *SIAM J. Appl. Math.* 36 (1979):273-80.
9. Ch. A. Charalambides & M. Koutras. "On the Differences of ghe Generalized Factorials at an Arbitrary Point and Their Combinatorial Applications." *Discrete Mathematics* 47 (1983):183-201.
10. N. L Johnson & S. Kotz. *Urn Models and Their Application.* New York: Wiley, 1977.
11. J. Riordan. *An Introduction to Combinatorial Analysis.* New York: Wiley, 1958.

◆◇◆◇◆