ON A FAMILY OF NESTED RECURRENCES

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1. INTRODUCTION

A recursive definition of a function f is called *nested* if, in the definition body, the function f is called with an argument whose evaluation involves yet another call to function f. A famous example of such a nested recursive definition is "McCarthy's 91-function"

$$f(n) = \begin{cases} f(f(n + 11)) & 0 \leqslant n \leqslant 100 \\ n - 10 & n > 100 \end{cases}$$

whose solution

$$f(n) = \begin{cases} 91 & 0 \leqslant n \leqslant 100 \\ n - 10 & n > 100 \end{cases}$$

is described in [2, p. 373]. Such recurrences seem difficult to understand and solve, and general solution techniques are lacking.

In this paper a complete solution is developed for the family of nested recurrences (one for each integer $k > 0$) given by

$$g_k(n) = \begin{cases} n - g_k(g_k(n - k)) & n \geqslant 1 \\ 0 & n \leqslant 0. \end{cases} \tag{1.1}$$

For the case $k = 1$, this recurrence is mentioned in [1, p. 137], where its behavior is described diagramatically.

The functions $g_1(n)$ and $g_2(n)$ are plotted in Figures 1 and 2.

Recently Meek and van Rees [3] have examined the recurrence family

$$f_r(n) = n - f_r(f_r(\cdots(f_r(n - 1))\cdots)),\ n \geqslant 1$$

where f_r is nested to r levels and $f_r(0) = 0$. In [3] the solution for $f_r(n)$ is expressed indirectly through a transformation: n is represented as a generalized Fibonacci base numeral (dependent on r), the least significant digit of this representation is truncated, and the resulting Fibonacci base numeral represents $f_r(n)$. In this paper we give a closed form solution for $f_2(n)$, which is $g_1(n)$ in our notation. The problem of finding a closed form for $f_r(n)$, $r \geqslant 3$, remains open.

The approximate behavior of $g_k(n)$ is easy to describe. Figures 1 and 2 suggest looking for an asymptotic approximation to the solution having the form $g_k(n) = An + O(1)$. Substituting this into (1.1) and equating coefficients of n on both sides yields

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$$g_k(n) = \psi n + O(1) \text{ as } n \to \infty, \tag{1.2}$$

where $\psi = (\sqrt{5} - 1)/2$ is the reciprocal of the "golden ratio" ϕ.* The relationship between $g_k(n)$ and the line ψn is even closer than the asymptotic estimate (1.2) would indicate, since Theorem 1 states

$$g_k(n) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n+i}{k} \right\rfloor + \psi \right\rfloor. \tag{1.3}$$

Thus, $g_1(n) = \lfloor \psi(n+1) \rfloor$ is the function described in [1]. This function has an interesting number-theoretic property: Theorem 2 shows that the points at which $g_1(n)$ increases form a Beatty sequence.

2. SOLUTION

Let us first give a solution for the function $g_1(n)$. From it, we generalize the solution of (1.1).

Figure 1 shows that while the line ψn must miss all the integral lattice points, the values of $g_1(n)$ fall on lattice points near the line. This behavior suggests looking for a solution of the form $g_1(n) = \lfloor \psi n + C \rfloor$. If one substitutes this form into (1.1) with $k = 1$ and performs calculations similar to those in Lemma 1 below, it emerges that a choice of $C = \psi$ will cause the equation to balance. Turning this calculation around into a proof yields the following Lemma, which shows that $g_1(n) = \lfloor \psi n + \psi \rfloor$. This result is also needed in the proof of Theorem 1.

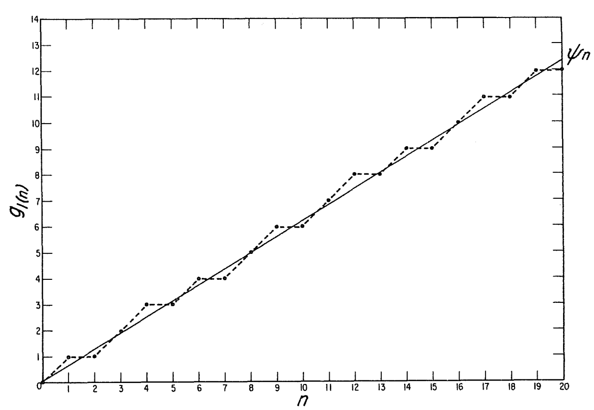

FIG. 1. Plot of $g_1(n)$ for $0 \leqslant n \leqslant 20$. The values of the function are indicated by heavy dots. The dashed lines are present only to facilitate interpretation. Superimposed on the function is the straight line ψn.

Lemma 1

For all $n \geqslant 0$, $\lfloor \psi n + \psi \rfloor = n - \lfloor \psi \lfloor \psi n \rfloor + \psi \rfloor$. (2.1)

*ϕ *is the positive root of* $\phi^2 - \phi - 1 = 0$, *while* $\psi = \phi^{-1} = \phi - 1$ *is the positive root of* $\psi^2 + \psi - 1 = 0$.

Proof: Let $\psi n = \lfloor \psi n \rfloor + \varepsilon$, where $\varepsilon = \psi n \bmod 1$, the fractional part of ψn. First, we note that ε can never equal ψ^2. For suppose $\psi n = \lfloor \psi n \rfloor + \psi^2$ for some n. Then $\psi n - \psi^2 = \psi n - (1 - \psi) = \psi(n + 1) - 1$ is an integer, and $\psi(n + 1)$ is an integer for some n, which is clearly impossible.

Now (2.1) is equivalent to the assertion

$$\lfloor \psi n + \psi \rfloor + \lfloor \psi(\psi n - \varepsilon) + \psi \rfloor = n.$$

This is equivalent to

$$\lfloor \psi n + \psi \rfloor + \lfloor \psi^2 n - \psi\varepsilon + \psi \rfloor = n.$$

Since $\psi^2 n = n - \psi n$, we may cancel the integer n, yielding

$$\lfloor \psi n + \psi \rfloor + \lfloor -\psi n - \psi\varepsilon - \psi \rfloor = 0,$$

or

$$\lfloor \lfloor \psi n \rfloor + \varepsilon + \psi \rfloor + \lfloor -\lfloor \psi n \rfloor - \varepsilon - \psi\varepsilon + \psi \rfloor = 0.$$

Cancelling the integers from inside the floor functions, this is equivalent to

$$\lfloor \varepsilon + \psi \rfloor + \lfloor \psi - \varepsilon(1 + \psi) \rfloor = 0.$$

This last identity can be seen to hold for all $\varepsilon \neq \psi^2$ in the interval $(0, 1)$ as follows: The argument of the second floor term is linear in ε, decreasing over $(0, 1)$, with a zero at $\varepsilon = \psi^2$. In case $\varepsilon < \psi^2$, both terms yield zero, because the arguments of each floor are positive and less than 1. In case $\varepsilon > \psi^2$, the first term is 1 and the second is -1. ∎

Next, we turn to the solution of g_2, defined by $g_2(n) = n - g_2(g_2(n - 2))$. At even arguments $n = 2m$, we have

$$g_2(2m) = 2m - g_2(g_2(2(m - 1))). \tag{2.2}$$

Define the function h via

$$g_2(2i) = 2h(i). \tag{2.3}$$

Then (2.2) can be written

$$2h(m) = 2m - g(2h(m - 1)) = 2m - 2h(h(m - 1)),$$

by using (2.3) again. Thus

$$h(m) = m - h(h(m - 1))$$

with $h(0) = 0$ and so $h(m) = g_1(m) = \lfloor \psi m + \psi \rfloor$. Putting this into (2.3) and using $n = 2m$ yields finally

$$g_2(n) = 2\left\lfloor \psi \frac{n}{2} + \psi \right\rfloor, \; n \text{ even}. \tag{2.4}$$

To solve for odd arguments n is not so straightforward. But an examination of Figure 2 shows that the values of $g_2(n)$ at odd n seem to lie on a straight line between the neighboring values at even arguments. This observation suggests that the solution is the "average" of the two nearest even argument values, or

$$g_2(n) = \left\lfloor \psi \left\lfloor \frac{n}{2} \right\rfloor + \psi \right\rfloor + \left\lfloor \psi \left\lfloor \frac{n + 1}{2} \right\rfloor + \psi \right\rfloor, \; n \geqslant 0.$$

This expression is certainly consistent with (2.4). That this is indeed the solution is established by an induction argument. In fact, the "natural" generalization of this expression, given by (2.7), will be shown in Theorem 1 to satisfy the general recurrence (1.1).

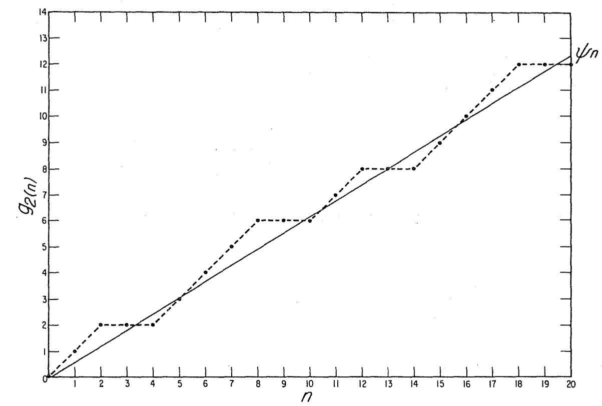

FIG. 2. Plot of $g_2(n)$ for $0 \leqslant n \leqslant 20$. The values of the function are indicated by heavy dots. The dashed lines are present only to facilitate interpretation. Superimposed on the function is the straight line ψn. This function appears to be a "scaled up" version of Figure 1.

The following lemma is needed for the induction of Theorem 1.

Lemma 2

For all $n \geqslant 0$, $0 \leqslant g_k(n) \leqslant n$. (2.5)

Proof: By induction on n. The base $0 \leqslant n < k$ is easily checked, since

$$g_k(n) = n$$

for arguments in this range. Assume that $n \geqslant k$ and that (2.5) holds for all $0 \leqslant i < n$. We will establish (2.5) for n. Now

$$g_k(n) = n - g_k(g_k(n - k)). \tag{2.6}$$

Let $i = g_k(n-k)$. By the induction hypothesis for $n-k$, we have $0 \leqslant i \leqslant n - k$, and so by the induction hypothesis for i, $0 \leqslant g_k(i) \leqslant i$, that is,

$$0 \leqslant g_k(g_k(n - k)) \leqslant g_k(n - k).$$

Using this inequality with (2.6) yields

$$n - g_k(n - k) \leqslant g_k(n) \leqslant n,$$

and, since $n - k - g_k(n - k) \geqslant 0$ by the induction hypothesis, the result (2.5) follows for n. ■

Now to the main result.

Theorem 1

The solution to (1.1) is given by

$$g_k(n) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor + \psi \right\rfloor, \; n \geqslant 0. \tag{2.7}$$

Proof: By induction on n. The base $0 \leqslant n < k$ can be checked directly, as $g_k(n) = n$ for arguments in this range. Assume that $n \geqslant k$ and that (2.7) holds for all $0 \leqslant i < n$. We will establish (2.7) for n.

By the induction hypothesis,

$$g_k(n - k) = \sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor \right\rfloor. \tag{2.8}$$

Suppose that $n = qk + r$ with remainder $0 \leqslant r < k$. Then the first $k - r$ of the quotients

$$\left\lfloor \frac{n}{k} \right\rfloor, \left\lfloor \frac{n + 1}{k} \right\rfloor, \ldots, \left\lfloor \frac{n + k - 1}{k} \right\rfloor$$

are equal to q and the remaining r quotients are equal to $q + 1$. Thus,

$$g_k(n - k) = (k - r)\lfloor \psi q \rfloor + r\lfloor \psi q + \psi \rfloor. \tag{2.9}$$

and similarly

$$\sum_{i=0}^{k-1} \left\lfloor \psi \left\lfloor \frac{n + i}{k} \right\rfloor + \psi \right\rfloor = (k - r)\lfloor \psi q + \psi \rfloor + r\lfloor \psi q + 2\psi \rfloor. \tag{2.10}$$

We would like to show that $g_k(n)$ is equal to (2.10). There are two cases to consider.

Case $\lfloor \psi q + \psi \rfloor = \lfloor \psi q \rfloor$: Then it follows that

$$\lfloor \psi q + 2\psi \rfloor = 1 + \lfloor \psi q + \psi \rfloor, \tag{2.11}$$

and by (2.9), $g_k(n - k) = k\lfloor \psi q \rfloor$. But then all the quotients

$$\left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor, \quad 0 \leqslant i < k \tag{2.12}$$

are identically equal to $\lfloor \psi q \rfloor$. Since $g_k(n - k) \leqslant n - k$ by Lemma 2, the induction hypothesis (2.7) holds with argument set to $g_k(n - k)$, and so using the equality of all the quotients (2.12)

$$g_k(g_k(n - k)) = k\lfloor \psi \lfloor \psi q \rfloor + \psi \rfloor. \tag{2.13}$$

By Lemma 1, the right side of (2.13) is $k(q - \lfloor \psi q + \psi \rfloor)$, and using this fact in (1.1):

$$g_k(n) = n - g_k(g_k(n - k)) = qk + r - g_k(g_k(n - k)) = r + k\lfloor \psi q + \psi \rfloor. \tag{2.14}$$

Using (2.11) in (2.10) gives agreement with the expression for $g_k(n)$ in (2.14), establishing the step in this case.

Case $\lfloor \psi q + \psi \rfloor = \lfloor \psi q \rfloor + 1$: In this case (2.9) yields

$$g_k(n - k) = k\lfloor \psi q \rfloor + r. \tag{2.15}$$

Because of (2.15), we obtain

$$\begin{aligned} \left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor &= \lfloor \psi q \rfloor, \qquad 0 \leqslant i < k - r \\ \left\lfloor \frac{g_k(n - k) + i}{k} \right\rfloor &= \lfloor \psi q \rfloor + 1, \; k - r \leqslant i < k \end{aligned} \tag{2.16}$$

Lemma 2 guarantees that $g_k(n-k) \leqslant n-k$, so the induction hypothesis (2.7) holds with argument $g_k(n-k)$. Along with identities (2.16), this gives

$$g_k(g_k(n-k)) = (k-r)\lfloor\psi\lfloor\psi q\rfloor + \psi\rfloor + r\lfloor\psi\lfloor\psi q\rfloor + 2\psi\rfloor,$$

which in light of the case assumption can be rewritten

$$g_k(g_k(n-k)) = (k-r)\lfloor\psi\lfloor\psi q\rfloor + \psi\rfloor + r\lfloor\psi\lfloor\psi(q+1)\rfloor + \psi\rfloor. \tag{2.17}$$

Now apply Lemma 1 to each of the terms in (2.17), and simplify to obtain

$$g_k(g_k(n-k)) = kq + r - (k-r)\lfloor\psi q + \psi\rfloor - r\lfloor\psi q + 2\psi\rfloor. \tag{2.18}$$

From this, using the recurrence (1.1),

$$g_k(n) = n - g_k(g_k(n-k)) = (k-r)\lfloor\psi q + \psi\rfloor + r\lfloor\psi q + 2\psi\rfloor. \tag{2.19}$$

and this is seen to be just (2.10), as required. This case completes the induction and the proof of (2.7). ■

3. THE DISTRIBUTION OF TRANSITION POINTS FOR $g_1(n)$

Let

$$\nabla f(n) = f(n) - f(n-1),\ n = 1, 2, 3, \ldots$$

be the "backward difference" sequence of the function $f(n)$, $n = 0, 1, 2, \ldots$. The values of n for which $\nabla f(n) \neq 0$ are called the *transition points* of f and the sequence T_f of the values of n for which $\nabla f(n) \neq 0$ is called the *transition sequence* for f.

Successive values for $g_1(n)$ clearly can differ by at most one. That is, ∇g_1 is a sequence of zeros and ones. As observed in Figure 1, the distribution of transition points for $g_1(n)$ also shows considerable regularity. In fact, Theorem 2 establishes that T_{g_1} is the Beatty sequence [4, pp. 29-30] for the "golden ratio" $\phi = \psi + 1$.

Beatty sequences are defined as follows: if α and β are positive irrationals such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

then the two sequences

$$B_\alpha = \{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \ldots\} \quad \text{and} \quad B_\beta = \{\lfloor\beta\rfloor, \lfloor 2\beta\rfloor, \lfloor 3\beta\rfloor, \ldots\}$$

are mutually exclusive and together contain all the positive integers without repetition. A proof may be found in [5, §12.2].

If $\alpha = \phi$, then $\beta = \phi + 1$ and the two complementary sequences are

$$B_\phi = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, \ldots\}$$

and

$$B_{\phi+1} = \{2, 5, 7, 10, 13, 15, 18, \ldots\}.$$

In order to show that $T_{g_1} = B_\phi$, we establish the following identities. The first states that the function $g_1(n)$ is the inverse of Beatty's function $\lfloor\phi n\rfloor$, and that transitions do occur at points in the sequence B_ϕ.

<u>Lemma 3</u>

$$g_1(\lfloor\phi n\rfloor) = n$$
$$g_1(\lfloor\phi n\rfloor - 1) = n - 1$$

Proof: $\lfloor \phi n \rfloor = \lfloor (\psi + 1)n \rfloor = \lfloor \psi n \rfloor + n$, hence

$$g_1(\lfloor \phi n \rfloor) = g_1(\lfloor \psi n \rfloor + n) = \lfloor \psi(\lfloor \psi n \rfloor + n + 1) \rfloor$$

by Theorem 1. Using $\psi n = \lfloor \psi n \rfloor + \varepsilon$,

$$g_1(\lfloor \phi n \rfloor) = \lfloor \psi^2 n + \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor n + \psi(1 - \varepsilon) \rfloor = n,$$

where we have used $\psi^2 + \psi = 1$.

For the second identity, note that

$$\lfloor \phi n \rfloor - 1 = \lfloor (\psi + 1)n - 1 \rfloor = \lfloor \psi n \rfloor + n - 1$$

so that

$$g_1(\lfloor \phi n \rfloor - 1) = \lfloor \psi(\lfloor \psi n \rfloor + n) \rfloor = \lfloor \psi^2 n + \psi n - \psi(\varepsilon) \rfloor$$
$$= \lfloor n - \psi(\varepsilon) \rfloor = n - 1. \blacksquare$$

Lemma 4

Let $\varepsilon = \psi n \bmod 1$ be the fractional part of ψn. Then for all n,

$$\lfloor \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor \psi n \rfloor = \lfloor \psi n - \psi \varepsilon \rfloor. \tag{3.1}$$

Proof: Obviously $0 < \psi(1 - \varepsilon) < 1 - \varepsilon$, and so

$$\psi n < \psi n + \psi(1 - \varepsilon) < \psi n + 1 - \varepsilon.$$

Since $\psi n = \lfloor \psi n \rfloor + \varepsilon$,

$$\lfloor \psi n \rfloor + \varepsilon < \psi n + \psi(1 - \varepsilon) < \lfloor \psi n \rfloor + 1,$$

and so it follows that $\lfloor \psi n \rfloor = \lfloor \psi n + \psi(1 - \varepsilon) \rfloor$, establishing the first equality.

Next, notice that

$$\psi n - \psi \varepsilon = \lfloor \psi n \rfloor + \varepsilon(1 - \psi).$$

Since $0 < \varepsilon(1 - \psi) < 1$, the second equality follows. $\blacksquare$

The next lemma gives information about the points where g_1 does not have a transition.

Lemma 5

$$g_1(\lfloor (\phi + 1)n \rfloor) = g_1(\lfloor (\phi + 1)n \rfloor - 1) = \lfloor \phi n \rfloor. \tag{3.2}$$

Proof: Consider the first equality. Since $\phi = \psi + 1$, by Theorem 1 this is equivalent to showing that

$$\lfloor \psi \lfloor \psi n + 2n \rfloor + \psi \rfloor = \lfloor \psi \lfloor \psi n + 2n \rfloor \rfloor. \tag{3.3}$$

Now (3.3) is equivalent to showing

$$\lfloor \psi \lfloor \psi n \rfloor + 2n\psi + \psi \rfloor = \lfloor \psi \lfloor \psi n \rfloor + 2n\psi \rfloor. \tag{3.4}$$

Let $\psi n = \lfloor \psi n \rfloor + \varepsilon$ where $\varepsilon = \psi n \bmod 1$. Substituting this into (3.4) and simplifying using $\psi^2 + \psi = 1$ shows that (3.4) is equivalent to

$$\lfloor n + \psi n + \psi(1 - \varepsilon) \rfloor = \lfloor n + \psi n - \psi \varepsilon \rfloor. \tag{3.5}$$

By Lemma 4, these expressions are equal, proving that (3.3) holds.

Consider the second equality. By Theorem 1 this is equivalent to showing that

$$\lfloor \psi \lfloor \psi n + 2n \rfloor \rfloor = \lfloor \psi n + n \rfloor, \tag{3.6}$$

which is equivalent to

$$\lfloor \psi \lfloor \psi n \rfloor + 2\psi n \rfloor = \lfloor \psi n + n \rfloor. \tag{3.7}$$

Using the substitution $\lfloor \psi n \rfloor = \psi n - \varepsilon$ in (3.7), and simplifying using $\psi^2 + \psi = 1$ shows this is equivalent to

$$\lfloor n + \psi n - \psi \varepsilon \rfloor = \lfloor \psi n + n \rfloor. \tag{3.8}$$

By Lemma 4, this last equality holds, and so (3.6) holds. ■

The connection with the Beatty sequence can now be made.

Theorem 2

$T_{g_1} = B_\phi$.

Proof: By Lemma 3,

$$g_1(\lfloor \phi n \rfloor) - g_1(\lfloor \phi n \rfloor - 1) = 1$$

so that $\lfloor \phi n \rfloor$ are transition points corresponding to B_ϕ, while by Lemma 5

$$g_1(\lfloor (\phi + 1)n \rfloor) - g_1(\lfloor (\phi + 1) \rfloor n - 1) = 0,$$

so that the nontransition points $\lfloor (\phi + 1)n \rfloor$ correspond to $B_{\phi+1}$. By the properties of Beatty sequences, B_ϕ and $B_{\phi+1}$ include all the positive integers. ■

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