PELL AND PELL-LUCAS POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to record some properties of *Pell polynomials* $P_n(x)$ and *Pell-Lucas polynomials* $Q_n(x)$ defined by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Initially, the polynomials are defined for $n \geqslant 0$ but their existence for $n < 0$ is readily extended, yielding

$$P_{-n}(x) = (-1)^{n+1}P_n(x) \tag{1.3}$$

and

$$Q_{-n}(x) = (-1)^n Q_n(x). \tag{1.4}$$

Some of these polynomials are:

$$\begin{cases} P_2(x) = 2x, \quad P_3(x) = 4x^2 + 1, \quad P_4(x) = 8x^3 + 4x, \\ P_5(x) = 16x^4 + 12x^2 + 1, \quad P_6(x) = 32x^5 + 32x^3 + 6x, \ \ldots; \end{cases} \tag{1.5}$$

$$\begin{cases} Q_2(x) = 4x^2 + 2, \quad Q_3(x) = 8x^3 + 6x, \quad Q_4(x) = 16x^4 + 16x^2 + 2, \\ Q_5(x) = 32x^5 + 40x^3 + 10x, \quad Q_6(x) = 64x^6 + 96x^4 + 36x^2 + 2, \ \ldots\ . \end{cases} \tag{1.6}$$

Important special numerical cases are: $P_n(1) = P_n$, the n^{th} *Pell number*; $Q_n(1) = Q_n$, the n^{th} *Pell-Lucas number*; $P_n(\frac{1}{2}) = F_n$, the n^{th} *Fibonacci number*; and $Q_n(\frac{1}{2}) = L_n$, the n^{th} *Lucas number*. Furthermore, $P_n(\frac{1}{2}x) = F_n(x)$, the n^{th} *Fibonacci polynomial*, and $Q_n(\frac{1}{2}x) = L_n(x)$, the n^{th} *Lucas polynomial* (see [2]).

Following standard procedures, we easily obtain the *Binet forms*

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.7}$$

and

$$Q_n(x) = \alpha^n + \beta^n, \tag{1.8}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.9}$$

are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0, \tag{1.10}$$

so that

$$\alpha + \beta = 2x,\ \alpha - \beta = 2\sqrt{x^2 + 1},\ \ \alpha\beta = -1. \tag{1.11}$$

The generating functions for the infinite sets of polynomials $\{P_n(x)\}$ and $\{Q_n(x)\}$ are found in the usual way to be

$$\sum_{r=0}^{\infty} P_{r+1}(x)y^r = \frac{1}{1 - 2xy - y^2} \tag{1.12}$$

and

$$\sum_{r=0}^{\infty} Q_{r+1}(x)y^r = \frac{2x + 2y}{1 - 2xy - y^2}. \tag{1.13}$$

Results involving these generating functions are not developed here.

2. ELEMENTARY PROPERTIES OF $P_n(x)$, $Q_n(x)$

Important elementary relationships involving $P_n(x)$ and $Q_n(x)$ follow without difficulty with the aid of (1.7)-(1.11). Some of these are:

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x) = 2xP_n(x) + 2P_{n-1}(x) \tag{2.1}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2 + 1)P_n(x) \tag{2.2}$$

$$P_n(x)Q_n(x) = P_{2n}(x) \tag{2.3}$$

$$Q_{2n}(x) = \tfrac{1}{2}\{Q_n^2(x) + 4(x^2 + 1)P_n^2(x)\} \tag{2.4}$$

$$\left.\begin{aligned} P_{n+1}(x)P_{n-1}(x) - P_n^2(x) &= (-1)^n & (2.5)\\ Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) &= (-1)^{n-1}4(x^2 + 1) & (2.6)\end{aligned}\right\} \textit{Simson formulas}$$

$$P_{n+1}^2(x) - P_{n-1}^2(x) = 2xP_{2n}(x) \quad \text{by (1.1), (2.1), (2.3)} \tag{2.7}$$

$$4(x^2 + 1)P_n^2(x) - Q_n^2(x) = 4(-1)^{n-1} \tag{2.8}$$

Formula (2.3) is useful in establishing divisibility properties of the polynomials. Geometrical paradoxes can be constructed from (2.5) when numerical values of x are inserted.

Summations of an elementary nature are obtained in the usual manner. The simplest are:

$$\sum_{r=1}^{n} P_{2r}(x) = (P_{2n+1}(x) - 1)/2x \tag{2.9}$$

$$\sum_{r=1}^{n} P_{2r-1}(x) = P_{2n}(x)/2x \tag{2.10}$$

$$\sum_{r=1}^{n} P_r(x) = (P_{n+1}(x) + P_n(x) - 1)/2x \quad \text{by (2.9), (2.10)} \tag{2.11}$$

$$\sum_{r=1}^{n} Q_{2r}(x) = (Q_{2n+1}(x) - 2x)/2x \tag{2.12}$$

$$\sum_{r=1}^{n} Q_{2r-1}(x) = (Q_{2n}(x) - 2)/2x \tag{2.13}$$

$$\sum_{r=1}^{n} Q_r(x) = (Q_{n+1}(x) + Q_n(x) - 2 - 2x)/2x \quad \text{by (2.12), (2.13)} \tag{2.14}$$

Extensions and variations of these finite summations, e.g., $\sum_{r=1}^{n} r^2P_r(x)$ and $\sum_{r=1}^{n}(-1)^r Q_r(x)$, are omitted in this treatment of the polynomials.

Induction can be used, with a little effort, to establish the explicit expressions

$$P_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \binom{n-m-1}{m}(2x)^{n-2m-1} \tag{2.15}$$

and

$$Q_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-m}\binom{n-m}{m}(2x)^{n-2m}, \quad n \neq 0, \tag{2.16}$$

where, in (2.16) we used the combinatorial identity

$$\frac{n}{n-m}\binom{n-m}{m} + \frac{n-1}{n-m}\binom{n-m}{m-1} = \frac{n+1}{n-m+1}\binom{n-m+1}{m}.$$

We proceed to prove (2.15).

Proof of (2.15): The formula is trivially true for $n = 1$ and $n = 2$. Assume it is true for $n = k$ and $n = k - 1$ where $k \geqslant 3$. Then we have

$$\begin{aligned} P_{k+1}(x) &= 2xP_k(x) + P_{k-1}(x) \quad \text{by (1.1)} \\ &= \sum_{m=0}^{\left[\frac{k-1}{2}\right]} \binom{k-m-1}{m}(2x)^{k-2m} + \sum_{m=0}^{\left[\frac{k-2}{2}\right]} \binom{k-m-2}{m}(2x)^{k-2m-2}. \end{aligned}$$

If $k = 2t$, this becomes

$$\begin{aligned} &\sum_{m=0}^{t-1} \binom{2t-m-1}{m}(2x)^{2t-2m} + \sum_{m=0}^{t-1} \binom{2t-m-2}{m}(2x)^{2t-2m-2} \\ &= \binom{2t-1}{0}(2x)^{2t} + \binom{2t-2}{1}(2x)^{2t-2} + \binom{2t-3}{2}(2x)^{2t-4} + \cdots + \binom{t}{t-1}(2x)^2 \\ &\quad + \binom{2t-2}{0}(2x)^{2t-2} + \binom{2t-3}{1}(2x)^{2t-4} + \cdots + \binom{t}{t-2}(2x)^2 + \binom{t-1}{t-1} \\ &= \sum_{m=0}^{t} \binom{2t-m}{m}(2x)^{2t-2m} = \sum_{m=0}^{[k/2]} \binom{k-m}{m}(2x)^{k-2m} \end{aligned}$$

by using Pascal's formula. Similarly, it holds if k is odd, and the proof is completed.

Basic relationships involving $P_n(x)$ and $Q_n(x)$ may be obtained from these combinatorial formulas, but the calculations required are tedious. Binet forms produce the same results more quickly.

In passing, we note the differential calculus result:

$$\frac{dQ_n(x)}{dx} = 2nP_n(x). \tag{2.17}$$

Later, in (6.20), we shall see that the first derivative of $P_n(x)$ is given in terms of a (complex) Gegenbauer polynomial.

Because $P_n(x)$ and $Q_n(x)$ are generalizations of F_n and L_n, the collection of miscellaneous results for F_n and L_n given in [7] may be generalized; e.g.,

$$Q_{4n}(x) - 2 = 4(x^2 + 1)P_{2n}^2(x), \tag{2.18}$$

$$P_{n-1}(x)P_{n+1}(x) + Q_{n-1}(x)Q_{n+1}(x) = (4x^2 + 5)P_n^2(x) + (-1)^{n-1}(4x^2 - 1), \tag{2.19}$$

and

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} P_{2k+p}(x) = [4(x^2+1)]^n Q_{2n+p+1}(x). \tag{2.20}$$

3. MATRIX GENERATION OF FORMULAS

We demonstrate that the matrix

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

generates Pell polynomials and Pell-Lucas polynomials, and use it to establish some elementary properties of these polynomials.

Induction, with (1.1), leads to

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \tag{3.2}$$

whence

$$\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.3}$$

and

$$P_n(x) = [1 \quad 0]P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.4}$$

The characteristic equation of P is

$$\lambda^2 - 2x\lambda - 1 = 0 \tag{3.5}$$

with eigenvalues

$$\begin{cases} \alpha = x + \sqrt{x^2+1} \\ \beta = x - \sqrt{x^2+1} \end{cases} \tag{3.6}$$

By the division algorithm for polynomials,

$$\lambda^n = (\lambda^2 - 2x\lambda - 1)f(\lambda) + m\lambda + k, \tag{3.7}$$

where $f(\lambda)$ is of degree $n-2$ in λ and m, k are functions of x.

Put $\lambda = \alpha$ in (3.7). Then

$$\alpha^n = m\alpha + k. \tag{3.8}$$

Similarly,

$$\beta^n = m\beta + k. \tag{3.9}$$

Solving (3.8) and (3.9) yields

$$m = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad k = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}. \tag{3.10}$$

From (3.8)

$$P^n = mP + kI. \tag{3.11}$$

Equate the top right elements in (3.11) to obtain $m = P_n(x)$ so that the Binet form (1.7) for $P_n(x)$ is again produced from (3.10).

Use of (2.1) gives

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix} \tag{3.12}$$

and

$$Q_n(x) = [1 \quad 0]P^{n-1}\begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.13}$$

To illustrate the matrix technique, we prove

$$P_{m+n}(x) = P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) \tag{3.14}$$

for

$$\begin{aligned} P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) &= [P_m(x), \quad P_{m-1}(x)]\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [P_m(x), \quad P_{m-1}(x)]P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3)} \\ &= [1 \quad 0]P^{m+n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3) and } P^mP^n = P^{m+n} \\ &= P_{m+n}(x) \quad \text{by (3.4).} \end{aligned}$$

Similarly

$$Q_{m+n}(x) = P_{m-1}(x)Q_n(x) + P_m(x)Q_{n+1}(x). \tag{3.15}$$

From (3.14) and (3.15) with (3.2) and (3.12), we derive

$$\begin{bmatrix} P_{n+r}(x) \\ P_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.16}$$

and

$$\begin{bmatrix} Q_{n+r}(x) \\ Q_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.17}$$

Equation (3.14), including an interchange of m and n, in conjunction with (2.1) gives

$$P_{m+n}(x) = \tfrac{1}{2}\{P_m(x)Q_n(x) + P_n(x)Q_m(x)\}, \tag{3.18}$$

while (3.15), including a replacement of m by $m + 1$ and n by $n - 1$, with (2.1) and (2.2) gives

$$Q_{m+n}(x) = \tfrac{1}{2}\{Q_m(x)Q_n(x) + 4(x^2+1)P_m(x)P_n(x)\}. \tag{3.19}$$

Putting $m = n$ in (3.18) and (3.19) yields (2.3) and (2.4). Further,

$$P_{n+1}^2(x) + P_n^2(x) = P_{2n+1}(x) \tag{3.20}$$

since

$$\begin{aligned} P_{n+1}^2(x) + P_n^2(x) &= [P_{n+1}(x), \quad P_n(x)]\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [1 \quad 0]P^{2n}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{by (3.2) and (3.3)} \\ &= P_{2n+1}(x) \quad \text{by 3.4.} \end{aligned}$$

Result (3.20) also follows directly from (3.14) with $m = n + 1$.

Similarly,

$$Q_{n+1}^2(x) + Q_n^2(x) = 4(x^2+1)P_{2n+1}(x). \tag{3.21}$$

All the above results can, of course, be derived by using the Binet forms (1.7) and (1.8). Techniques employed in these sections give rise to the following formulas:

$$P_{n+r}(x) + P_{n-r}(x) = \begin{cases} P_n(x)Q_r(x) & \text{if } r \text{ is even} \\ Q_n(x)P_r(x) & \text{if } r \text{ is odd} \end{cases} \tag{3.22}$$

$$Q_{n+r}(x) + Q_{n-r}(x) = \begin{cases} Q_n(x)Q_r(x) & r \text{ even} \\ 4(x^2+1)P_n(x)P_r(x) & r \text{ odd} \end{cases} \tag{3.23}$$

$$P_{n+r}(x) - P_{n-r}(x) = \begin{cases} Q_n(x)P_r(x) & r \text{ even} \\ P_n(x)Q_r(x) & r \text{ odd} \end{cases} \tag{3.24}$$

$$Q_{n+r}(x) - Q_{n-r}(x) = \begin{cases} 4(x^2+1)P_n(x)P_r(x) & r \text{ even} \\ Q_n(x)Q_r(x) & r \text{ odd} \end{cases} \tag{3.25}$$

$$P_{n+r}^2(x) - P_{n-r}^2(x) = P_{2n}(x)P_{2r}(x) \quad \text{by (3.22), (3.24) and (2.3)} \tag{3.26}$$

$$Q_{n+r}^2(x) - Q_{n-r}^2(x) = 4(x^2+1)P_{2n}(x)P_{2r}(x) \quad \text{by (3.23), (3.25), and (2.3)} \tag{3.27}$$

$$P_{mn+r}(x) = \begin{cases} P_n(x)Q_{(m-1)n+r}(x) + (-1)^n P_{(m-2)n+r}(x) \\ P_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1}P_{(m-2)n+r}(x) \end{cases} \tag{3.28}$$

$$Q_{mn+r}(x) = Q_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1}Q_{(m-2)n+r} \tag{3.29}$$

$$\left.\begin{aligned} P_n^2(x) - P_{n+r}(x)P_{n-r}(x) &= (-1)^{n-r}P_r^2(x) & (3.30) \\ Q_n^2(x) - Q_{n+r}(x)Q_{n-r}(x) &= (-1)^{n-r+1}4(x^2+1)P_r^2(x) & (3.31) \end{aligned}\right\} \textit{Simson formulas}$$

$$P_{n+h}(x)P_{n+k}(x) - P_n(x)P_{n+h+k}(x) = (-1)^n P_h(x)P_k(x) \tag{3.32}$$

$$Q_{n+h}(x)Q_{n+k}(x) - Q_n(x)Q_{n+h+k}(x) = (-1)^{n-1}4(x^2 + 1)P_h(x)P_k(x) \tag{3.33}$$

$$P_{n+h}(x)Q_{n+k}(x) - P_n(x)Q_{n+h+k}(x) = (-1)^n P_h(x)Q_k(x) \tag{3.34}$$

Finally, we offer two relationships that can be described as being of the *de Moivre type*:

$$\{Q_n(x) + 2\sqrt{x^2+1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) + 2\sqrt{x^2+1}P_{nr}(x)\} \tag{3.35}$$

and

$$\{Q_n(x) - 2\sqrt{x^2+1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) - 2\sqrt{x^2+1}P_{nr}(x)\}. \tag{3.36}$$

When $x = \frac{1}{2}$, (3.35) and (3.36) reduce to

$$\left\{\frac{L_n + \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} + \sqrt{5}F_{nr}}{2} \tag{3.37}$$

and

$$\left\{\frac{L_n - \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} - \sqrt{5}F_{nr}}{2}, \tag{3.38}$$

respectively, the first of which is given in [7, p. 60].

Results involving $P_n(x)$ and $Q_n(x)$ are as multitudinous as the sands of the seashore, and one can gather these grains *ad infinitum, ad nauseam.*

4. PASCAL ARRAYS GENERATING $P_n(x)$, $Q_n(x)$

Consider the following table.

Table 1: Pell Polynomials from Rising Diagonals

$n \backslash m$	1	2	3	4	5	6 ...
1	1					
2	$2x$	1				
3	$4x^2$	$4x$	1			
4	$8x^3$	$12x^2$	$6x$	1		
5	$16x^4$	$32x^3$	$24x^2$	$8x$	1	
6	$32x^5$	$80x^4$	$80x^3$	$40x^2$	$10x$	1
⋮						

(4.1)

Denote the coefficient of the power of x in the m^{th} row and n^{th} column by (m, n).

It is now shown that the rising diagonals presented in Table 1 produce the Pell polynomial (1.5).

Define the entries in row m as the terms in the expansion $(2x+1)^{m-1}$, that is

$$\sum_{n=1}^{m} (m, n)x^{m-n} = (2x+1)^{m-1} \qquad m \geqslant n. \tag{4.2}$$

Hence,

$$(m,\ n) = \binom{m-1}{m-n}2^{m-n} \qquad m \geqslant n. \tag{4.3}$$

Now the rising diagonal function $R_m(x)$ of degree m in x in Table 1 is:

$$\begin{aligned} R_m(x) &= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} (m + 1 - n,\ n)x^{m+1-2n} \qquad (m \geqslant 1) &\quad (4.4)\\ &= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} \binom{m-n}{m+1-2n}(2x)^{m+1-2n} \qquad \text{by } (4.3)\\ &= \sum_{n=1}^{\left[\frac{m+1}{2}\right]} \binom{m-n}{n-1}(2x)^{m+1-2n}\\ &= \sum_{n=0}^{\left[\frac{m-1}{2}\right]} \binom{m-n-1}{n}(2x)^{m-1-2n}\\ &= P_m(x) \qquad \text{from } (2.15) \end{aligned}$$

Now consider Table 2.

Table 2: Pell-Lucas Polynomials from Rising Diagonals

n \ m	1	2	3	4	5	6	7 ...
1	$\cancel{2x}$	$\cancel{2}$					
2	$\cancel{4x^2}$	$\cancel{6x}$	$\cancel{2}$				
3	$\cancel{8x^3}$	$\cancel{16x^2}$	$\cancel{10x}$	$\cancel{2}$			
4	$\cancel{16x^4}$	$\cancel{40x^3}$	$\cancel{36x^2}$	$\cancel{14x}$	2		
5	$\cancel{32x^5}$	$\cancel{96x^4}$	$\cancel{112x^3}$	$64x^2$	$18x$	2	
6	$\cancel{64x^6}$	$\cancel{224x^5}$	$320x^4$	$240x^3$	$100x^2$	$22x$	2
⋮							

(4.5)

Let $[m,\ n]$ denote the coefficient of the power of x in the m^{th} row and n^{th} column.

We may define the entries in row m as the terms in the expansion of

$$(2x + 1)^m + (2x + 1)^{m-1} = (2x + 1)^{m-1}(2x + 2),$$

that is,

$$\sum_{n=1}^{m+1} [m,\ n]x^{m+1-n} = (2x + 1)^{m-1}(2x + 2) \tag{4.6}$$

and so

$$[m, n] = 2(m, n) + 2(m, n - 1) = 2(m, n) + (m, n - 1) + (m, n - 1)$$
$$= (m + 1, n) + (m, n - 1). \tag{4.7}$$

Denote the rising diagonal function of degree m in x in Table 2 by $S_m(x)$. Then

$$\begin{aligned} S_m(x) &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} [m + 1 - n, n]x^{m+2-2n} \\ &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} \{(m + 2 - n, n) + (m + 1 - n, n - 1)\}x^{m+2-2n} && \text{by (4.7)} \\ &= \sum_{n=1}^{\left[\frac{m+2}{2}\right]} \left\{\binom{m+1-n}{n-1} + \binom{m-n}{n-2}\right\}(2x)^{m+2-2n} && \text{by (4.3)} \\ &= \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m}{m-n}\binom{m-n}{n}(2x)^{m-2n} && \text{on simplification} \\ &= Q_m(x) && \text{by (2.16)} \end{aligned}$$

Thus, we have demonstrated that Pell and Pell-Lucas polynomials are generated by the rising diagonals in Table 1 and Table 2, respectively.

Next, arrange the coefficients of the powers of x in $P_n(x)$, (1.5), in the following Pascal-like display.

Table 3: Pell Polynomial Coefficients

Coeffs. in $P_n(x)$ \ Powers	0	1	2	3	4	5	6	7	8	9 ...
1	1									
2	0	2								
3	1	0	4							
4	0	4	0	8						
5	1	0	12	0	16					
6	0	6	0	32	0	32				
7	1	0	24	0	80	0	64			
8	0	8	0	80	0	192	0	128		
9	1	0	40	0	240	0	448	0	256	
10	0	10	0	160	0	672	0	1024	0	512
⋮										

Designate the entry in the r^{th} row and c^{th} column of Table 3 by $\{r, c\}$. From the table and (2.15), we have:

$$\{2r, 2c\} = 0 \tag{4.8}$$

$$\{2r,\ 2c-1\} = \begin{cases} \binom{r+c-1}{r-c}2^{2c-1} & c = 1, 2, \ldots, r \\ 0 & c > r \end{cases} \tag{4.9}$$

$$\{2r-1,\ 2c-1\} = 0 \tag{4.10}$$

$$\{2r-1,\ 2c\} = \begin{cases} \binom{r+c-1}{r-c-1}2^{2c} & c = 0, 1, 2, \ldots, r-1 \\ 0 & c \geqslant r \end{cases} \tag{4.11}$$

Using (4.8)-(4-11), we can prove:

$$\sum_{i=0}^{r-1}\{2r-1-i,\ i\} = 3^{r-1} \tag{4.12}$$

$$\sum_{i=1}^{2r}\{i,\ 2c-1\} = \tfrac{1}{2}\{2r+1,\ 2c\} \tag{4.13}$$

$$\sum_{i=1}^{2r}\{i,\ 2c\} = \tfrac{1}{2}\{2r,\ 2c+1\} \tag{4.14}$$

$$\sum_{i=1}^{2r-1}\{i,\ 2c-1\} = \tfrac{1}{2}\{2r-1,\ 2c\} \tag{4.15}$$

$$\sum_{i=1}^{2r-1}\{i,\ 2c\} = \tfrac{1}{2}\{2r,\ 2c+1\} \tag{4.16}$$

Proof of (4.12)

$$\begin{aligned}\sum_{i=0}^{r-1}\{2r-1-i,\ i\} &= \{2r-1,\ 0\} + \{2r-2,\ 1\} + \cdots + \{r,\ r-1\} \\ &= \binom{r-1}{r-1}2^0 + \binom{r-1}{r-2}2^1 + \cdots + \binom{r-1}{0}2^{r-1} \quad \text{by (4.9) and (4.11)} \\ &= (1+2)^{r-1} = 3^{r-1}\end{aligned}$$

Proof of (4.13)

$$\begin{aligned}\sum_{i=1}^{2r}\{i,\ 2c-1\} &= \{2,\ 2c-1\} + \{4,\ 2c-1\} + \cdots + \{2r,\ 2c-1\} \quad \text{by (4.10)} \\ &= \{2c,\ 2c-1\} + \{2c+2,\ 2c-1\} + \cdots + \{2r,\ 2c-1\} \quad \text{by (4.9)} \\ &= 2^{2c-1}\left(\binom{2c-1}{0} + \binom{2c}{1} + \cdots + \binom{r+c-1}{r-c}\right) \quad \text{by (4.9)} \\ &= 2^{2c-1}\left(\binom{2c-1}{2c-1} + \binom{2c}{2c-1} + \cdots + \binom{r+c-1}{2c-1}\right) \\ &= 2^{2c-1}\binom{r+c}{2c} \quad \text{by identity (1.52) in [6]} \\ &= \tfrac{1}{2}\{2r+1,\ 2c\} \quad \text{by (4.11)}\end{aligned}$$

If a similar table for $Q_n(x)$ is constructed, and if we designate the element in row r and column c by $\overline{r,\ c}$, we have from (2.1) that

$$\overline{r,\ c} = \{r+1,\ c\} + \{r-1,\ c\} = 2\{r,\ c-1\} + 2\{r-1,\ c\}. \tag{4.17}$$

Properties of $\overline{r,\ c}$ may then be developed on the basis of (4.8)-(4.11).

From (2.2), we derive

$$\overline{r+1,\ c} + \overline{r-1,\ c} = 4\{r,\ c\} + 4\{r,\ c-2\}. \tag{4.18}$$

To conclude this section, we establish a relationship between $(m,\ n)$ and $\{r,\ c\}$ in Tables 1 and 3, respectively (both relating to the Pell polynomials). A relationship between $[m,\ n]$ and $\overline{r,\ c}$ will also be formulated for the Pell-Lucas polynomials.

Now in (4.9), $2c-1$ is the power of x in $P_{2r}(x)$. Comparing the coefficient of the term x^{2c-1} in (2.15) with that in (4.3), where we recall that

$$\binom{m-1}{m-n} = \binom{m-1}{n-1}$$

we deduce that

$$\{2r,\ 2c-1\} = (r+c,\ r-c+1) \tag{4.19}$$

and so

$$(r,\ c) = \{r+c-1,\ r-c\}. \tag{4.20}$$

A similar argument applied to (2.15) and (4.3) for (4.1) yields

$$\{2r-1,\ 2c\} = (r+c,\ r-c)$$

whence (4.20) results again.

Lastly, consider $\overline{2r,\ 2c}$, the coefficient of x^{2c} in $Q_{2r}(x)$. From (4.17),

$$\overline{2r,\ 2c} = \left(\binom{r+c}{r-c} + \binom{r+c-1}{r-c-1}\right)2^{2c}.$$

Using (4.7) with (4.3), we find

$$[m,\ n] = \left(\binom{m}{n-1} + \binom{m-1}{n-2}\right)2^{m-n+1}$$

whence, by comparison of the two forms,

$$\overline{2r,\ 2c} = [r+c,\ r-c+1]. \tag{4.21}$$

Reversely,

$$[r,\ c] = \overline{r+c-1,\ r-c+1}. \tag{4.22}$$

A similar formula to (4.21) is

$$\overline{2r-1,\ 2c+1} = [r+c,\ r-c]$$

whence (4.22) results again.

5. DETERMINANTAL GENERATION OF $P_n(x)$, $Q_n(x)$

Write d_{ij} for the element in the i^{th} row and j^{th} column of an $n \times n$ determinant.

Let $\Delta_n(x)$ be the $n \times n$ determinant defined by

$$\Delta_n(x):\ \begin{cases} d_{ii} = 2x & i = 1,\ 2,\ \ldots,\ n \\ d_{i,\,i+1} = 1 & i = 1,\ \ldots,\ n-1 \\ d_{i,\,i-1} = -1 & i = 2,\ \ldots,\ n \\ d_{ij} = 0 & \text{otherwise} \end{cases} \tag{5.1}$$

From $\Delta_n(x)$, the determinants $\delta_n(x)$, $\Delta_n^*(x)$, and $\delta_n^*(x)$ are defined as follows:

$\delta_n(x)$: as for $\Delta_n(x)$ except that $d_{i,i+1} = -1$, $d_{i,i-1} = 1$ (5.2)

$\Delta_n^*(x)$: as for $\Delta_n(x)$ except that $d_{12} = 2$, $d_{i,i+1} = 1$ $(i = 2, \ldots, n-1)$ (5.3)

$\delta_n^*(x)$: as for $\Delta_n(x)$ except that $d_{12} = -2$, $d_{i,i+1} = -1$, $d_{i,i-1} = 1$. $(i = 2, \ldots, n-1)$ (5.4)

Induction and expansion along the first row, together with basic properties of $P_n(x)$ and $Q_n(x)$, e.g., (1.1), (2.1), yield

$$\Delta_n(x) = P_{n+1}(x) \tag{5.5}$$

$$\delta_n(x) = P_{n+1}(x) \tag{5.6}$$

$$\Delta_n^*(x) = Q_n(x) \tag{5.7}$$

$$\delta_n^*(x) = Q_n(x). \tag{5.8}$$

In the process of expansion, we derive recurrence relations such as

$$\Delta_k(x) = 2x\Delta_{k-1}(x) + \Delta_{k-2}(x) \qquad k \geqslant 3 \tag{5.9}$$

and

$$\Delta_k^*(x) = 2x\Delta_{k-1}^*(x) + 2\Delta_{k-2}^*(x) \qquad k \geqslant 3. \tag{5.10}$$

6. RELATIONS OF $P_n(x)$, $Q_n(x)$ TO OTHER FUNCTIONS

Perhaps the simplest results relating $P_n(x)$ to other functions are found in [4]:

$$\left.\begin{aligned} P_{2n}(x) &= \sinh 2nt/\cosh t & (6.1)\\ P_{2n+1}(x) &= \cosh(2n+1)t/\cosh t & (6.2)\end{aligned}\right\} x = \sinh t$$

Hence

$$\left.\begin{aligned} Q_{2n}(x) &= 2\cosh 2nt & (6.3)\\ Q_{2n+1}(x) &= 2\sinh(2n+1)t & (6.4)\end{aligned}\right\} x = \sinh t$$

Comparison of the explicit summation formulas for $P_n(x)$ and $Q_n(x)$ given in (2.15) and (2.16) with the explicit summation formulas for $U_n(x)$ and $T_n(x)$, the Chebyshev polynomials of the second and first kinds, respectively (see [11]), shows that

$$P_n(x) = (-i)^{n-1}U_{n-1}(ix) \tag{6.5}$$

and

$$Q_n(x) = 2(-i)^n T_n(ix) \tag{6.6}$$

i.e., $P_n(x)$ and $Q_n(x)$ are modified Chebyshev polynomials in a complex variable. To reconcile the form in [11] with (2.16) we had to replace the Gamma function, namely, $\Gamma(n-m) = (n-m-1)!$

Because of (6.5) and (6.6), $P_n(x)$ and $Q_n(x)$ would have [9] complex hypergeometric representations. Other representations also exist in view of the many forms the expressions for $U_n(x)$ and $T_n(x)$ can take.

In particular, we may record that

$$P_n(i \cosh x) = i^{n-1}\sinh nx/\sinh x \tag{6.7}$$

and

$$Q_n(i \cosh x) = 2i^n \cosh nx. \tag{6.8}$$

From (1.1) we observe that

$$P_{n+1}(ix) + P_{n-1}(ix) = Q_n(ix)$$

leads, with the help of (6.5) and (6.6), to

$$U_n(ix) - U_{n-2}(ix) = 2T_n(ix), \tag{6.9}$$

which is a complex version of a basic relationship between the two kinds of Chebyshev polynomials. Similarly, other Chebyshev relationships may be tied to corresponding relationships involving $P_n(x)$ and $Q_n(x)$.

Finally, we allude to the *Gegenbauer* (*ultraspherical*) *polynomial* of degree n and order ν, $C_n^\nu(x)$, defined by

$$\sum_{n=0}^{\infty} C_n^\nu(x)t^n = (1 - 2xt + t^2)^{-\nu} \qquad (\nu > 0, \quad |t| < 1). \tag{6.10}$$

with explicit forms

$$C_n^0(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r}{n - r}\binom{n-r}{r}(2x)^{n-2r} \qquad C_0^0(x) = 1 \quad (\nu = 0) \tag{6.11}$$

and

$$C_n^\nu(x) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{\Gamma(n - r + \nu)}{\Gamma(n - r + 1)}\binom{n-r}{r}(2x)^{n-2r} \qquad (\nu > -\tfrac{1}{2};\ \nu \neq 0). \tag{6.12}$$

A recurrence relation for $C_n^\nu(x)$ is

$$(n + 2)C_{n+2}^\nu(x) = 2(n + \nu + 1)xC_{n+1}^\nu(x) - (n + 2\nu)C_n^\nu(x) \tag{6.13}$$

which, for $\nu = 1$, reduces to

$$C_{n+2}^1(x) = 2xC_{n+1}^1(x) - C_n^1(x) \tag{6.14}$$

with

$$C_0^1(x) = 1, \quad C_1^1(x) = 2x. \tag{6.15}$$

Clearly, $C_n^1(x) = U_n(x)$, and by (6.5),

$$P_n(x) = (-i)^{n-1}C_{n-1}^1(ix). \tag{6.16}$$

When $\nu = 0$, (6.11), where $C_1^0(x) = 2x$, gives

$$C_n^0(x) = \frac{2}{n} T_n(x),$$

so that (6.6) gives

$$Q_n(x) = n(-i)\ C_n^0(ix) \qquad (n \geqslant 1) \tag{6.17}$$

i.e., $P_n(x)$, $Q_n(n)$ are modified Gegenbauer polynomials in a complex variable.

As the Fibonacci and Lucas numbers arise from $P_n(x)$ and $Q_n(x)$ when $x = \tfrac{1}{2}$, we have, from (6.16) and (6.17),

$$F_1 = C_0^1\left(\frac{i}{2}\right) = 1, \quad F_n = (-i)^{n-1}C_{n-1}^1\left(\frac{i}{2}\right) \tag{6.18}$$

and

$$L_0 = 2C_0^0\left(\frac{i}{2}\right) = 2, \quad L_n = n(-1)^n C_n^0\left(\frac{i}{2}\right) \qquad n \geqslant 1. \tag{6.19}$$

Using the known [9] result $dT_n(x)/dx = nU_{n-1}(x)$ from [11] with (6.5) and (6.6), we can arrive back at (2.17), viz., $dQ_n(x)/dx = 2nP_n(x)$.

Differentiating in (2.15) and applying (6.12) in the case $\nu = 2$, we deduce that

$$\frac{dP_n(x)}{dx} = 2(-i)^{n-2}C^2_{n-2}(ix). \tag{6.20}$$

Alternatively, we may differentiate in (6.16) and invoke the result [11]

$$\frac{dC^\nu_n(x)}{dx} = 2\nu C^{\nu+1}_{n-1}(x)$$

to obtain (6.20).

Some of the above results, e.g., (6.16), were generalized in [12] for the sequence of polynomials $\{A_k(x)\}$ defined by

$$A_{n+2}(x) = 2xA_{n+1}(x) + A_n(x) \qquad A_0(x) = s, \quad A_1(x) = r. \tag{6.21}$$

Of course, $\{A_n(x)\}$ is a special case of the sequence $\{W_n(p, q; a, b)\}$, some of whose properties are documented in [8].

Information related to some aspects of the above ideas can be found in [1], [2], [3], [4], [5], [9], and [10].

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