ON $P_{r,k}$ SEQUENCES

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(Submitted May 1983)

INTRODUCTION

Definition 1: Let k be a given positive integer. Two integers α and β are said to have the property p_k (resp. p_{-k}) if $\alpha\beta + k$ (resp. $\alpha\beta - k$) is a perfect square. A set of integers is said to be a P_k set if every pair of distinct elements in the set has the property p_k. A sequence of integers is said to be a $P_{r,k}$ sequence if every r consecutive terms of the sequence constitute a P_k set.

Given a positive integer k, we can always find two integers α and β having the property p_k. Conversely, given two integers α and β, we can always find a positive integer k such that α and β have the property p_k. If S is a given P_k set and j is a given integer, then by multiplying all the elements of S by j, we obtain a P_{kj^2} set. Suppose we are given two numbers $a_1 < a_2$ with property p_k and we want to extend the set $\{a_1, a_2\}$ such that the resulting set is also a P_k set. Toward this end, in this paper we construct a $P_{3,k}$ sequence $\{a_n\}$.

ASSOCIATED $P_{3,k}$ SEQUENCES

Suppose

$$a_1a_2 + k = b_1^2 \tag{1}$$

and let $a_3 \in \{a_1, a_2, \ldots\}$, a P_k set. Then we have

$$a_1a_3 + k = x^2 \tag{2}$$

and

$$a_2a_3 + k = y^2 \tag{3}$$

for some integers x and y. Eliminating a_3 from (2) and (3), we obtain

$$X^2 - a_1a_2Y^2 = ka_2(a_2 - a_1), \tag{4}$$

where $X = a_2x$, $Y = y$. Using (1) in (4), we obtain

$$X^2 - (b_1^2 - k)Y^2 = k(a_2^2 - b_1^2 + k). \tag{5}$$

One can check that $X = a_2(a_1 + b_1)$, $Y = a_2 + b_1$, is always a solution of (5). When $b_1^2 - k$ is positive and square free, (5) has an infinite number of solutions. Henceforth, we concentrate on the solution $X = a_2(a_1 + b_1)$, $Y = a_2 + b_1$ of (5). This gives

$$a_2a_3 + k = b_2^2,$$
$$a_1a_3 + k = c_1^2,$$

with

AMS (MOS) Subject Classifications (1980) 10 B 05.

$$b_2 = a_2 + b_1, \quad c_1 = a_1 + b_1, \quad a_3 = b_2 + c_1.$$

In what follows, we construct three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, where a_1, a_2, a_3, b_1, b_2, and c_1 are as above. We say that $\{b_n\}$ and $\{c_n\}$ are the sequences associated with $\{a_n\}$. Taking

$$b_3 = a_3 + b_2, \quad c_2 = a_2 + b_2, \quad a_4 = b_3 + c_2,$$

we can see that $2(a_3 + a_2) - a_1 = a_4$. Using this fact, we obtain

$$a_2a_4 + k = c_2^2 \quad \text{and} \quad a_3a_4 + k = b_3^2.$$

For the construction of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, the following diagram can be helpful.

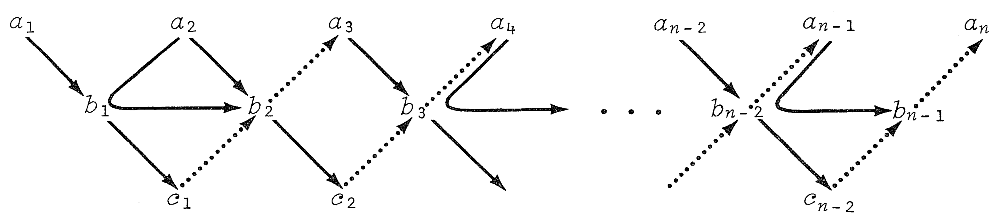

Diagram 1

Explanation for the diagram: Write $b_1 = \sqrt{a_1a_2 + k}$ in the second row, in the space between a_1 and a_2; and write $c_1 = \sqrt{a_1a_3 + k}$ in the third row, in the space beneath a_2. Along the arrows shown by thick lines, sum the elements of of the first and second rows to obtain the elements of the third row. Along the curved arrows, sum the elements of the first and second rows to obtain the elements of the second row. Along the arrows shown by dotted lines, sum the elements of the second and third rows to obtain the elements of the first row. The preceding discussion shows that the scheme provided in the diagram is valid for a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, and c_2. Let $n > 2$. Assuming the validity of Diagram 1 for $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$, and $c_1, \ldots, c_{n-2}$, it can be proved without much difficulty that

$$2(a_n + a_{n-1}) - a_{n-2} = a_{n+1}, \tag{6}$$

and that the scheme is valid for $a_1, \ldots, a_{n+1}, b_1, \ldots, b_n$, and $c_1, \ldots, c_{n-1}$.

Theorem 1. The three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have the same recurrence relation.

Proof: We have $a_{n+1} = 2(a_n + a_{n-1}) - a_{n-2}$ [see (6)]. Now

$$\begin{aligned} b_{n+1} &= a_{n+1} + b_n = c_{n-1} + 2b_n = a_{n-1} + b_{n-1} + 2b_n \\ &= 2b_n + b_{n-1} + (b_{n-1} - b_{n-2}) = 2(b_n + b_{n-1}) - b_{n-2}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} c_{n+1} &= a_{n+1} + b_{n+1} = 2a_{n+1} + b_n = 2(c_{n-1} + b_n) + b_n \\ &= 2c_{n-1} + b_n + 2(c_n - a_n) = 2(c_n + c_{n-1}) + (a_n + b_{n-1}) - 2a_n \\ &= 2(c_n + c_{n-1}) - c_{n-2}. \end{aligned} \tag{8}$$

Hence, the theorem is proved.

We shall now obtain additional relations. First, using

$$a_{n+1} = c_{n+1} - b_{n+1} \quad \text{and} \quad a_{n+2} = c_n + b_{n+1},$$

we have

$$a_{n+1} + a_{n+2} = c_n + c_{n+1};$$

that is,

$$a_{n+1} - c_n = -(a_{n+2} - c_{n+1}). \tag{9}$$

Next, from

$$b_n = c_n - a_n \quad \text{and} \quad b_n = b_{n+1} - a_{n+1},$$

we obtain

$$2b_n = (c_n + b_{n+1}) - a_{n+1} - a_n,$$

which yields

$$2b_n = a_{n+2} - a_{n+1} - a_n. \tag{10}$$

Next,

$$\begin{aligned} a_{n+2} - a_{n+1} + a_n &= (b_{n+1} + c_n) - (b_{n+1} - b_n) + a_n \\ &= c_n + b_n + a_n \\ &= 2c_n. \end{aligned} \tag{11}$$

From (10), we obtain

$$a_{n+2} = a_{n+1} + a_n + 2\sqrt{a_n a_{n+1} + k},$$

and from (6) we have

$$a_{n+2} = 2(a_{n+1} + a_n) - a_{n-1}.$$

Hence,

$$a_{n+1} + a_n - a_{n-1} = 2\sqrt{a_n a_{n+1} + k},$$

which gives the relation

$$a_{n+1}^2 + a_n^2 + a_{n-1}^2 - 2a_{n-1}a_n - 2a_{n-1}a_{n+1} - 2a_n a_{n+1} = 4k. \tag{12}$$

FIBONACCI RELATIONSHIPS

Next we shall exhibit a relationship between either of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ and the Fibonacci sequence $\{F_n\}$. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

V. E. Hoggatt, Jr., and G. E. Bergum [1] have shown that the even-subscripted Fibonacci numbers constitute a $P_{3,1}$ sequence. We can set

$$a_{n-1} = F_{2n}, \quad a_n = F_{2n+2}, \quad \text{and} \quad a_{n+1} = F_{2n+4}$$

in (12) and obtain

$$F_{2n}^2 + F_{2n+2}^2 + F_{2n+4}^2 - 2F_{2n}F_{2n+2} - 2F_{2n+2}F_{2n+4} - 2F_{2n}F_{2n+4} = 4.$$

Theorem 2. Any sequence $\{a_n\}$ satisfying (6) is given by

$$a_n = -F_{n-3}F_{n-2}a_1 + F_{n-3}F_{n-1}a_2 + F_{n-2}F_{n-1}a_3, \quad n \geqslant 4. \tag{13}$$

Proof: From (6), we get

$$a_4 = 2(a_3 + a_2) - a_1 = -F_1F_2a_1 + F_1F_3a_2 + F_2F_3a_3,$$
$$a_5 = 2(a_4 + a_3) - a_2 = -2a_1 + 3a_2 + 6a_3 = -F_2F_3a_1 + F_2F_4a_2 + F_3F_4a_3,$$
$$a_6 = 2(a_5 + a_4) - a_3 = -6a_1 + 10a_2 + 15a_3 = -F_3F_4a_1 + F_3F_5a_2 + F_4F_5a_3.$$

So the theorem is true for $n = 4, 5, 6$. Let $n \geqslant 4$ and assume that the theorem is true for all integers j up to n. Using (6) we have

$$\begin{aligned} a_{n+1} = {} & 2(-F_{n-3}F_{n-2}a_1 + F_{n-3}F_{n-1}a_2 + F_{n-2}F_{n-1}a_3) \\ & + 2(-F_{n-4}F_{n-3}a_1 + F_{n-4}F_{n-2}a_2 + F_{n-3}F_{n-2}a_3) \\ & - (-F_{n-5}F_{n-4}a_1 + F_{n-5}F_{n-3}a_2 + F_{n-4}F_{n-3}a_3); \end{aligned}$$

that is,

$$\begin{aligned} a_{n+1} = {} & (-2F_{n-3}F_{n-2} - 2F_{n-4}F_{n-3} + F_{n-5}F_{n-4})a_1 \\ & + (2F_{n-3}F_{n-1} + 2F_{n-4}F_{n-2} - F_{n-5}F_{n-3})a_2 \\ & + (2F_{n-2}F_{n-1} + 2F_{n-3}F_{n-2} - F_{n-4}F_{n-3})a_3. \end{aligned} \tag{14}$$

The coefficient of a_1 in (14) is given by

$$\begin{aligned} -[2F_{n-3}(F_{n-2} + F_{n-4}) - F_{n-4}(F_{n-3} - F_{n-4})] &= -(2F_{n-3}F_{n-2} + F_{n-3}F_{n-4} + F_{n-4}^2) \\ &= -(2F_{n-3}F_{n-2} + F_{n-4}F_{n-2}) \\ &= -F_{n-2}(2F_{n-3} + F_{n-4}) \\ &= -F_{n-2}(F_{n-3} + F_{n-2}) \\ &= -F_{n-2}F_{n-1}. \end{aligned}$$

Similarly, upon simplification, we have the coefficients of a_2 and a_3 in (14) equal to $F_{n-2}F_n$ and $F_{n-1}F_n$, respectively. This proves Theorem 2.

Remark 1. The relations (6), (7), and (8) imply that (13) remains true if the a's are replaced by b's or by c's.

Now we express b's in terms of a_1, a_2, a_3. We have

$$2b_2 = -a_1 + a_2 + a_3.$$

Using $a_4 = 2(a_3 + a_2) - a_1$, we obtain

$$2b_3 = -a_2 + a_3 + a_4 = -a_1 + a_2 + 3a_3,$$
$$2b_4 = -a_2 + a_3 + 3a_4 = -3a_1 + 5a_2 + 7a_3.$$

Suppose $2b_n = -r_na_1 + s_na_2 + t_na_3$. Then

$$2b_{n+1} = -r_na_2 + s_na_3 + t_na_4 = -t_na_1 + 2(t_n - r_n)a_2 + (2t_n + s_n)a_3.$$

Hence, $2b_{n+1} = -r_{n+1}a_1 + s_{n+1}a_2 + t_{n+1}a_3$, where

$$\begin{aligned} & t_2 = 1, \quad t_3 = 3, \quad t_4 = 7, \\ & r_{n+1} = t_n, \\ & s_{n+1} = 2t_n - t_{n-1}, \\ & t_{n+1} = 2(t_n + t_{n-1}) - t_{n-2} \qquad (n \geqslant 4). \end{aligned} \tag{15}$$

Similarly, we have $2c_{n+1} = -u_{n+1}a_1 + v_{n+1}a_2 + w_{n+1}a_3$, where

$$\begin{aligned} &w_1 = w_2 = 1, \quad w_3 = 5, \\ &u_{n+1} = w_n, \\ &v_{n+1} = 2w_n - w_{n-1}, \\ &w_{n+1} = 2(w_n + w_{n-1}) - w_{n-2} \qquad (n \geqslant 3). \end{aligned} \tag{16}$$

Thus, the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{t_n\}$, and $\{w_n\}$ have the same recurrence relation.

Next we consider the possibility for the coincidence of the sequences $\{a_n\}$ and $\{c_n\}$. In this regard, we have the following:

Theorem 3. Let $\{a_n\}$ be a $P_{3,k}$ sequence with the associated sequences $\{b_n\}$ and $\{c_n\}$. The following statements are equivalent:

(i)	$a_{n+1} = c_n$	for some integer $n \geqslant 1$
(ii)	$a_{n+1} = c_n$	for all integers n
(iii)	$a_{n+1} = b_n + c_n$	for all integers n
(iv)	$c_{n+1} = b_{n+1} + c_n$	for all integers n
(v)	$a_{n+1} = a_n + b_n$	for all integers n
(vi)	$b_{n+2} = 3b_{n+1} - b_n$	for all integers n
(vii)	$c_{n+2} = 3c_{n+1} - c_n$	for all integers n
(viii)	$a_{n+2} = 3a_{n+1} - a_n$	for all integers n
(ix)	$k = a_{n+1}^2 - 3a_n a_{n+1} + a_n^2$	for all integers n
(x)	$-k = b_{n+1}^2 - 3b_n b_{n+1} + b_n^2$	for all integers n
(xi)	$k = c_{n+1}^2 - 3c_n c_{n+1} + c_n^2$	for all integers n
(xii)	$a_n = -F_{2n-4}a_1 + F_{2n-2}a_2$	for all integers n
	and	
	$b_n = -F_{2n-3}a_1 + F_{2n-1}a_2$	for all integers $n \geqslant 3$
(xiii)	b_n is a $P_{3,-k}$ sequence with the associated sequences $\{a_n\}$ and $\{b_n\}$ (where $b_n b_{n+1} - k = a_{n+1}^2$).	

Proof: The following scheme may be adopted.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ii) $\Rightarrow$ (i),

(v) $\Rightarrow$ (ix) $\Rightarrow$ (viii), (v) $\Rightarrow$ (x) $\Rightarrow$ (vi);

(ii) $\Rightarrow$ (xi) $\Rightarrow$ (vii);

(ii) $\Rightarrow$ (xii) $\Rightarrow$ (ii) and (x) $\Rightarrow$ (xiii) $\Rightarrow$ (x).

The proof itself is left to the reader.

F-TYPE SEQUENCES

Definition 2: Let $\{a_n\}$ be a $P_{3,k}$ sequence together with associated sequences $\{b_n\}$ and $\{c_n\}$. We say that $\{a_n\}$ is an F-type sequence if the sequence

$$\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\},$$

obtained by juxtaposing the two sequences $\{a_n\}$ and $\{b_n\}$, is of Fibonacci type, i.e., $f_1 = a_1$, $f_2 = b_1$, and $f_n = f_{n-1} + f_{n-2}$, $n \geqslant 3$.

Theorem 4. A $P_{3,k}$ sequence $\{a_n\}$ with the associated sequences $\{b_n\}$ and $\{c_n\}$ for which any one of the equivalent statements in Theorem 3 holds is an F-type sequences. Conversely, given a Fibonacci-type sequence

$$T = \{g, h, g + h, g + 2h, \ldots\},$$

where g and h are two positive integers with $g < h$, if $\{a_n\}$ and $\{b_n\}$ are the sequences formed by taking the terms in the odd and even places, respectively, of T, in the same order as they appear in T, there is an integer k such that $\{a_n\}$ is an F-type $P_{3,k}$ sequence for which the equivalent statements in Theorem 3 hold.

Proof: ($\Rightarrow$) Using $c_{n-1} = a_{n-1} + b_{n-1}$, we obtain $a_n = a_{n-1} + b_{n-1}$ for $n \geqslant 2$. We have that $b_n = a_{n-1} + b_{n-1}$ for $n \geqslant 2$. Hence, the sequence $\{a_1, b_1, a_2, b_2, \ldots\}$ is of the Fibonacci type.

($\Leftarrow$) We have

$$a_1 = g, \quad b_1 = h,$$

$$a_n = F_{2n-3}g + F_{2n-2}h, \quad b_n = F_{2n-2}g + F_{2n-1}h \qquad (n \geqslant 2), \tag{17}$$

where $\{F_n\}$ is the Fibonacci sequence. One can check that

$$a_n + a_{n+2} = 3a_{n+1} \text{ for all } n \geqslant 1. \tag{18}$$

Now

$$\begin{aligned}
&(a_{n+2}^2 - 3a_{n+1}a_{n+2} + a_{n+1}^2) - (a_{n+1}^2 - 3a_n a_{n+1} + a_n^2) \\
&= (a_{n+2}^2 - a_n^2) - 3a_{n+1}(a_{n+2} - a_n) \\
&= (a_{n+2} - a_n)(a_{n+2} + a_n - 3a_{n+1}) = 0 \quad \text{for all } n \geqslant 1.
\end{aligned}$$

Hence, we have

$$a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = a_{n+2}^2 - 3a_{n+1}a_{n+2} + a_{n+1}^2 = \text{constant, for all } n.$$

Let $a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = k$. In particular, putting $n = 1$, we get

$$k = h^2 - gh - g^2. \tag{19}$$

We have, using (19),

$$\begin{aligned}
a_n a_{n+1} + k = (F_{2n-3}F_{2n-1} - 1)g^2 &+ (F_{2n-3}F_{2n} + F_{2n-2}F_{2n-1} - 1)gh \\
&+ (F_{2n-2}F_{2n} + 1)h^2.
\end{aligned}$$

It can be seen that $F_{2n-3}F_{2n} - 1 = F_{2n-2}F_{2n-1}$. Therefore,

$$a_n a_{n+1} + k = F_{2n-2}^2 g^2 + 2F_{2n-2}F_{2n-1}gh + F_{2n-1}^2 h^2 = b_n^2.$$

Next,

$$\begin{aligned}
a_{n-1}a_n + k = (F_{2n-5}F_{2n-1} - 1)g^2 &+ (F_{2n-5}F_{2n} + F_{2n-4}F_{2n-1} - 1)gh \\
&+ (F_{2n}F_{2n-4} + 1)h^2.
\end{aligned}$$

After some calculation, we have

$$a_{n-1}a_n + k = F_{2n-3}^2 g^2 + 2F_{2n-2}F_{2n-3}gh + F_{2n-2}^2 h^2 = a_n^2.$$

Consequently, the sequence $\{a_n\}$ is an F-type $P_{3,k}$ sequence with the associated c-sequence given by $c_n = a_{n+1}$ for all integers $n \geqslant 1$.

ASSOCIATED DIOPHANTINE EQUATIONS

Theorem 5. Given a positive integer k, an F-type $P_{3,k}$ sequence exists if and only if the Diophantine equation

$$x^2 - 5y^2 = 4k \tag{20}$$

is solvable in integers.

Proof: ($\Rightarrow$) Let $\{a_n\}$ be an F-type $P_{3,k}$ sequence with the associated sequence $\{b_n\}$ so that $\{a_1, b_1, a_2, b_2, \ldots\}$ is a sequence of the Fibonacci type wherein the relations are given by (17). Then

$$k = h^2 - gh - g^2;$$

that is,

$$h^2 - gh - (g^2 + k) = 0.$$

Treating this as a quadratic equation in h, we obtain $h = \dfrac{g \pm \sqrt{5g^2 + 4k}}{2}$. This implies

$$5g^2 + 4k = A^2$$

for some integer A. Hence, equation (20) is solvable in integers.

($\Leftarrow$) Let (x, y) be an integral solution of (20). Then $x \equiv y \pmod 2$. Form the Fibonacci-type sequence $\{a_1, b_1, a_2, b_2, \ldots\}$ by taking $a_1 = y$, $b_1 = (x + y)/2$. Then by Theorem 4 there is an integer k' such that $\{a_n\}$ is an F-type $P_{3,k'}$ sequence. We have $k' = a_2^2 - 3a_1a_2 + a_1^2$. Since

$$a_2 = a_1 + b_2 = \frac{x + 3y}{2}$$

we obtain

$$k' = \frac{x^2 - 5y^2}{4} = k.$$

Theorem 6. Given a positive integer k, a necessary condition for the existence of an F-type $P_{3,k}$ sequence is that

$$k \not\equiv 2, 3, 6, 7, 8, 10, 12, 13, 14, 17, 18 \pmod{20}$$

and

$$k \not\equiv 10, 15, 35, 40, 60, 65, 85, 90 \pmod{100}.$$

We omit the proof.

To prove our next result, we need the following:

Theorem 7. (Nagell [4]) If $u + v\sqrt{D}$ and $u' + v'\sqrt{D}$ are two given solutions of the equation

$$u^2 - Dv^2 = C \quad (D\text{: positive, square free}),$$

a necessary and sufficient condition for these two solutions to belong to the same class is that the two numbers $(uu' - vv'D)/C$ and $(vu' - uv')/C$ be integers.

In the following theorem, we prove a result for the Diophantine equation (20) by considering the terms of the corresponding F-type $P_{3,k}$ sequence.

Theorem 8. Given a positive integer k, the number of distinct classes of solutions of equation (20) is divisible by 3.

Proof: If (20) is not solvable in integers, then the theorem holds trivially. Assume the solvability of (20). Let (x_1, y_1) be an integral solution of (20). Take $a_1 = y_1$, $b_1 = (x_1 + y_1)/2$ and $a_2 = a_1 + b_1$; i.e., $a_2 = (x_1 + 3y_1)/2$. Then by Theorem 5 we have

$$k = a_2^2 - 3a_1a_2 + a_1^2$$

and $\{a_n\}$ is an F-type $P_{3,k}$ sequence. We have

$$b_2 = a_2 + b_1 = x_1 + 2y_1,$$

$$a_3 = a_2 + b_2 = \frac{3x_1 + 7y_1}{2}, \quad b_3 = a_3 + b_2 = \frac{5x_1 + 11y_1}{2},$$

$$a_4 = a_3 + b_3 = 4x_1 + 9y_1, \quad b_4 = a_4 + b_3 = \frac{13x_1 + 29y_1}{2}.$$

Choose x_i, y_i $(i = 2, 3, 4)$ such that $y_i = a_i$ and $(x_i + y_i)/2 = b_i$; i.e., $x_i = 2b_i - y_i$. Then $x_2 = (3x_1 + 5y_1)/2$, $x_3 = (7x_1 + 15y_1)/2$, $x_4 = (9x_1 + 20y_1)/2$. One can easily check that $x_i + \sqrt{5}y_i$ $(i = 2, 3, 4)$ are solutions of (20). Since

$$\frac{x_1y_2 - y_1x_2}{4k} = \frac{1}{2}, \quad \frac{x_1y_3 - y_1x_3}{4k} = \frac{3}{2}, \quad \text{and} \quad \frac{x_2y_3 - y_2x_3}{4k} = \frac{1}{2},$$

by Theorem 7 it follows that each $x_i + \sqrt{5}y_i$ $(i = 1, 2, 3)$ belongs to a distinct class of solutions of (20). Now

$$x_4 + \sqrt{5}y_4 = (9x_1 + 20y_1) + \sqrt{5}(4x_1 + 9y_1) = (x_1 + \sqrt{5}y_1)(9 + 4\sqrt{5})^n.$$

Since $9 + 4\sqrt{5}$ is the fundamental solution of the equation

$$u^2 - 5v^2 = 1,$$

it follows that $x_1 + \sqrt{5}y_1$ and $x_4 + \sqrt{5}y_4$ belong to the same class of solutions of (20). Thus, given a solution $x_1 + \sqrt{5}y_1$ of (20), we obtain three consecutive terms a_i $(i = 1, 2, 3)$ of an F-type $P_{3,k}$ sequence which in turn yield two more solutions $x_i + \sqrt{5}y_i$ $(i = 2, 3)$ of (20) such that $x_i + \sqrt{5}y_i$ $(i = 1, 2, 3)$ belong to different classes of solutions of (20). Further, it follows by simple induction that, for any integers i, i', j, the terms a_{3i+j} and $a_{3i'+j}$ $(j = 0, 1, 2)$ yield solutions of (20) which belong to the same class. Hence, every F-type $P_{3,k}$ sequence contributes exactly three distinct classes of solutions of (20). Consequently, the number of distinct classes of solutions of (20) is divisible by 3.

Definition 3: Given a positive integer k, two $P_{3,k}$ sequences $\{a_n\}$ and $\{a_n'\}$ are said to be distinct if there do not exist integers r and s such that $a_r = a_s'$.

Theorem 9. Given a positive integer k, the number of distinct F-type $P_{3,k}$ sequences is equal to 1/3 of the number of distinct classes of solutions of (20).

Proof: Follows from Theorem 8.

CONCLUDING COMMENTS

Our next investigation is on $P_{r,k}$ sequences with $r \geqslant 4$. Regarding this, we prove the following theorem.

Theorem 10. If $k \equiv 2 \pmod 4$, then there is no $P_{r,k}$ sequence with $r \geqslant 4$.

Proof: We follow the reasoning given by S. Mohanty [3]. Let $k \equiv 2 \pmod 4$ and let $\{a_n\}$ be a $P_{4,k}$ sequence. Then, for any two integers i, j satisfying $|j - i| \leqslant 3$, we have

$$a_i a_j + k = B^2 \tag{21}$$

for some integer B. If $a_i \equiv 0 \pmod 4$ or if $a_j \equiv 0 \pmod 4$, then (21) implies $B^2 \equiv 2 \pmod 4$, which is impossible. Hence, neither a_i nor a_j is 0 (mod 4). If $a_i \equiv a_j \pmod 4$, we have a contradiction; thus the elements a_i, a_{i+1}, a_{i+2}, and a_{i+3} do not share the property p_k.

The foregoing complements the work of Horadam, Loh, and Shannon [2], whose Pellian sequence $\{Q_n(N)\}$ is a $P_{3,N-2}$ sequence which is there also related to the even-subscripted Fibonacci numbers, to perfect squares, and to Diophantine equations.

ACKNOWLEDGMENT

The authors are extremely grateful to the referee for his helpful comments and suggestions.

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