A PATH COUNTING PROBLEM IN DIGRAPHS

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1. INTRODUCTION

In this paper, we consider only directed graphs without loops or multiple edges. Our terminology and notation will be standard except as noted. A good reference for any undefined terms is [1].

Our main goal is to determine the maximum possible number of directed paths between a pair of vertices in an acyclic digraph with m edges (and any number of vertices). Denoting this maximum possible number by $N(m)$, we will establish that

$$N(m) = \begin{cases} F_{(m+1)/2} & \text{for } m \text{ odd} \\ 1 & \text{for } m = 2 \\ 2F_{(m/2)-1} & \text{for } m \geqslant 4 \text{ and even} \end{cases}$$

where F satisfies the recurrence relation

$$F_k = F_{k-1} + F_{k-2},\ F_1 = 1,\ F_2 = 2.$$

The actual proof of this formula will be preceded by a sequence of five easy lemmas.

We then conclude with a brief discussion of the following related question: Given a positive integer k, what is the least number of edges in an acyclic digraph having *exactly* k directed paths between a pair of vertices.

2. PROOFS OF THE LEMMAS AND MAIN RESULT

Lemma 1

Let D be an acyclic digraph. Then D contains vertices a and z such that indegree a = outdegree $z = 0$. (We call a and z, respectively, a source and a sink of D.)

Proof: Let $x \in V(D)$. Consider a longest path directed away from x, say from x to z. Then outdegree $z = 0$ (since any edge leaving z would yield either a longer directed path away from x or a directed cycle in D).

The proof that indegree $a = 0$ for some $a \in V(D)$ is entirely analogous. ∎

Lemma 2

Let D be an acyclic digraph. Then the vertices of D can be ordered, say $1, 2, \ldots, n$, such that every edge in D is of the form (i, j), where $i < j$.

Proof: We proceed by induction on $n = |V(D)|$. The result is trivially true for $n = 2$. For the induction step, choose any $z \in V(D)$ with outdegree $z = 0$ (one exists by Lemma 1), and consider the digraph $D - z$. By the induction hypothesis, the vertices of $D - z$ can be ordered, say $1, 2, \ldots, n - 1$, in the manner described. If we let z be the n^{th} vertex, we have the desired ordering of $V(D)$. ■

In what follows, we assume D is an acyclic digraph with vertices ordered $1, 2, \ldots, n$ such that every edge of D is of the form (i, j), where $i < j$.

For any $x \in V(D)$, let $p_D(x)$ denote the number of directed paths from 1 to x in D. [When D is clear from context, we will use just $p(x)$ for this number.]

Lemma 3

Suppose D has a set of vertices $S = \{i < \cdots < j < k\}$, with $1 < i < k \leqslant n$, which induces a tournament (i.e., a digraph with every pair of vertices joined by precisely one edge). Then

$$p(k) \geqslant p(i) + \cdots + p(j).$$

Proof: For each $x \in S$, let $P(x)$ denote the set of directed paths from 1 to x. If $x \neq k$, let $P'(x)$ denote the set of directed paths from 1 to k obtained by taking a path from 1 to x together with the edges (x, k). Then, clearly,

$$P'(i) \cup \cdots \cup P'(j) \subseteq P(k),$$

and the sets on the left side are disjoint. Since

$$|P'(x)| = |P(x)| = p(x),$$

it follows at once that

$$p(i) + \cdots + p(j) \leqslant p(k). \ ■$$

Let $N(m)$ denote the maximum possible number of directed paths between two vertices of an acyclic digraph with m edges. Certainly $N(m)$ is a nondecreasing function of m. Let us call an acyclic digraph on m edges having precisely $N(m)$ directed paths between some pair of vertices a *path maximum m-graph*. It is easily seen that there will be a path maximum m-graph D with the vertices ordered as in Lemma 2 such that 1 and n are joined by precisely $N(m)$ directed paths, and 1 (resp., n) is the unique source (resp., sink) in D. We will assume this property for the path maximum m-graphs we consider in what follows.

Lemma 4

There exists a path maximum m-graph D in which

$$\{x \in V(D) \mid (x, n) \in E(D)\}$$

(i.e., the predecessors of n in D) induce a tournament.

Proof: Otherwise, let i, j be two predecessors of n (with say $i < j$) such that $(i, j) \notin E(D)$. Form the digraph

$$D' = D - (i, n) + (i, j).$$

To each directed path in D from 1 to n containing the edge (i, n) there corresponds uniquely a directly path in D' from 1 to n containing the edges (i, j) and (j, n). Hence, $p_{D'}(n) \geqslant p_D(n)$, and so D' is also a path maximum m-graph in which n has one less predecessor than in D. We simply iterate this procedure until we obtain a path maximum m-graph with the desired properties. ■

Lemma 5

If $m \geqslant 3$, there exists a path maximum m-graph in which n has indegree 2.

Proof: Let D be a path maximum m-graph in which the predecessors of n (ordered say $1 < \cdots < j < k$) induce a tournament. By Lemma 3,

$$p(k) \geqslant p(i) + \cdots + p(j).$$

Hence,

$$2p(k) \geqslant p(i) + \cdots + p(j) + p(k) = p(n) = N(m).$$

If indegree $n \geqslant 3$, we can construct a new acyclic digraph D' with m edges, as shown in Figure 1. Note that

$$p_{D'}(n') = 2p(k) \geqslant N(m),$$

and hence D' is also a path maximum m-graph. But indegree $_{D'} n' = 2$, and the proof is complete. ■

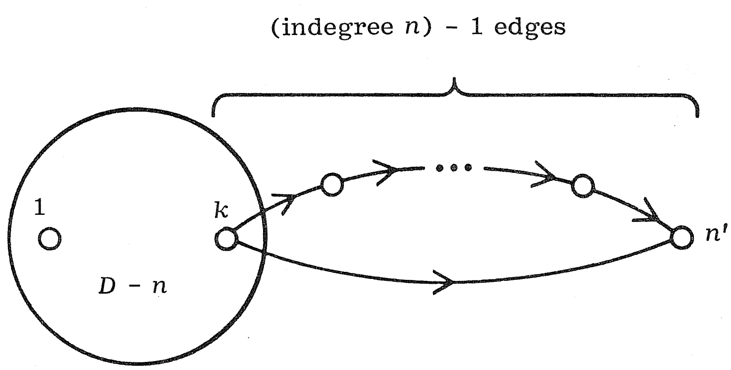

Figure 1. The Digraph D'

We now state and prove our main result.

Theorem

Let m be a positive inteter. Then

$$N(m) = \begin{cases} F_{(m+1)/2} & \text{for } m \text{ odd} \\ 1 & \text{for } m = 2 \\ 2F_{(m/2)-1} & \text{for } m \geqslant 4 \text{ and even} \end{cases}$$

where F_k is the Fibonacci number satisfying $F_k = F_{k-1} + F_{k-2}$, $F_1 = 1$, $F_2 = 2$.

Proof: It is readily verified that

$$N(1) = N(2) = 1,\ N(3) = N(4) = 2,\ N(5) = 3,\ N(6) = 4,$$

and so the formula holds for $m \geqslant 6$. We thus proceed by induction on $m \geqslant 7$.

Since the digraphs in Figure 2 contain m edges, and have as many dipaths from 1 to n as the number specified in the formula, it suffices to show the numbers in the formula are upper bounds for $N(m)$.

By Lemma 5 there is a path maximum m-graph D in which the indegree of n is 2. Let x, y denote the predecessors of n in D, with say $x < y$. We then have precisely three possibilities:

(i) $(x, y) \notin E(D)$ (Using the construction in the proof of Lemma 4, we could obtain a path maximum m-graph in which n has indegree 1.)

(ii) $(x, y) \in E(D)$, and x is the only predecessor of y.

(iii) $(x, y) \in E(D)$, and x is not the only predecessor of y.

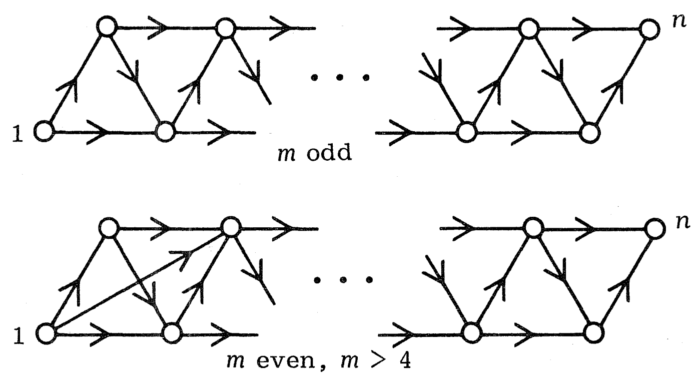

Figure 2. Path Maximum m-Graphs

By considering the maximum possible number of dipaths from the source to x and y in cases (i), (ii), and (iii), respectively, we get

$$N(m) \leqslant \max\{N(m - 1), 2N(m - 3), N(m - 2) + N(m - 4)\}.$$

Using the induction hypothesis, and the fact that $m \geqslant 7$, we obtain

$$N(m) \leqslant \begin{cases} \max\{2F_{(m-3)/2}, 4F_{(m-5)/2}, F_{(m-1)/2} + F_{(m-3)/2}\} = F_{(m+1)/2}, & \text{if } m \text{ odd}, \\ \max\{F_{(m/2)}, 2F_{(m/2)-1}, 2F_{(m/2)-2} + 2F_{(m/2)-3}\} = 2F_{(m/2)-1}, & \text{if } m \text{ even}. \end{cases}$$

The inductive step, and hence the proof of the theorem, are now complete. ∎

3. A RELATED PROBLEM

The authors have also considered the following problem: Given a positive integer k, what is the least number of edges in an acyclic digraph having *exactly* k paths between some pair of vertices? Noting the $N(m)$ is nondecreasing in m, it seems reasonable to conjecture that if $N(m - 1) < k \leqslant N(m)$, then m is the minimum number of edges required. This conjecture is indeed true for $k \leqslant 32$. However, $N(14) < 33 < N(15)$, and we have shown that at least 16 edges are needed in any digraph having exactly 33 directed paths between a pair of vertices. Although it appears that a complete solution to this problem may be very difficult, we have the following conjecture to offer:

Conjecture: Let k_n be the smallest integer such that $N(m - 1) < k_n < N(m)$, but at least $m + n$ edges are needed in any digraph with precisely k_n directed paths between a pair of vertices. Then k_n satisfies the recurrence relation $k_n = 34k_{n-1} + 21$, $k_1 = 33$.

REFERENCE

1. M. Behzad, G. Chartrand, & L. Lesniak-Foster. *Graphs and Digraphs*. Boston, Mass.: Prindle, Weber and Schmidt, 1979.

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