LATIN CUBES AND HYPERCUBES OF PRIME ORDER

CHARLES LAYWINE
Brock University, St. Catharines, Ontario L2S 3A1

GARY L. MULLEN
The Pennsylvania State University, University Park, PA 16802

(Submitted September 1983)

1. INTRODUCTION

In [3], the first author obtained an expression for the number of equivalence classes induced on the set of $n \times n$ Latin squares under row and column permutations. The first purpose of this paper is to point out that the results of [3] do not hold for all n, but rather that they hold only if n is a prime.* The second purpose of this paper is, in the case of prime n, to extend the results of [3] to three-dimensional and finally to n-dimensional Latin hypercubes. This is done in Sections 3 and 4.

2. LATIN SQUARES

A Latin square of order n is an $n \times n$ array with the property that each row and each column contains a permutation of the integers 1, 2, ..., n. In [3], two Latin squares were said to be equivalent if one could be obtained from the other by a permutation of the rows and another possibly different permutation of the columns, while a Latin square was said to be stationary if it remained invariant under some nontrivial row and column permutations. Let G be the group of all permutations of rows and columns so that G is isomorphic to $S_n \times S_n$ where S_n is the symmetric group on n letters. A Latin rectangle is an $m \times n$ array ($m \leqslant n$) in which each row contains a permutation of 1, 2, ..., n and no integer occurs more than once in any column. Denote the number of $m \times n$ Latin rectangles by $L(m, n)$.

---

*We now correct two errors that occur in [3]. In the proof of Lemma 1.2 of [3] it is assumed that if d divides n then the expression $L(kd+1, n)/L(kd, n)$ is always an integer for $k = 0, 1, \ldots, n/d - 1$. That this is not always the case is easily seen in the case when $n = 4$. Let $d = 2$ and $k = 1$, and consider $L(3, 4)/L(2, 4)$. It is easily checked (see, e.g., [2]) that $L(3, 4) = 4!3!4$, while $L(2, 4) = 4!9$, so that $L(3, 4)/L(2, 4) = 8/3$. Lemma 1.2 of [3] is corrected in our Lemma 1.2.

In Theorem 2 of [3], it is indicated that, if n is prime, then there are $(n - 2)!$ classes of stationary Latin squares each of which contains $(n - 1)! \times (n - 2)!$ elements. While the proof of the theorem is correct, the statement contains a typographical error and should read "For n prime, there are $(n - 2)!$ equivalence classes of stationary Latin squares, each of which contains $n! \times (n - 1)!$ elements."

It is now easy to prove

Lemma 1.2

Let $\Pi = (\Pi_r, \Pi_c)$ be a permutation of G such that both Π_r and Π_c consist of either p 1-cycles or 1 p-cycle, where p is a prime. Then there are either $L(p, p)$ or $L(1, p)$ Latin squares invariant under Π.

Proof: Clearly, if Π_r and Π_c both consist of p 1-cycles, then all Latin squares of order p are invariant while in the remaining case the first row can be chosen in $p! = L(1, p)$ ways. Once the first row is completed, the remaining rows are uniquely determined by Π_c.

We now prove

Theorem 1

If p is a prime, then permutations of rows and columns induce

$$\frac{L(p, p)}{(p!)^2} + \frac{(p-1)!}{p}$$

equivalence classes in the p^{th}-order Latin squares.

Proof: Burnside's lemma gives the number of classes as

$$(1/|G|)\sum_{\Pi \in G} \psi(\Pi)$$

where $\psi(\Pi)$ is the number of squares invariant under Π, from which the theorem follows.

It may be noted that, if ℓ_p denotes the number of reduced Latin squares of order p, then $L(p, p) = p!(p-1)!\ell_p$ so that the number of equivalence classes thus reduces to $(\ell_p + (p-1)!)/p$. Moreover, the values of ℓ_p are known if $p \leqslant 9$ (see [1]).

3. LATIN CUBES

In this section we extend the results of [3] to Latin cubes of prime order. A Latin cube C of order p is a $p \times p \times p$ array with the property that each of the p^3 elements c_{ijk} is one of the numbers 1, 2, ..., p and $\{c_{ijk}\}$ ranges over all of the numbers 1, 2, ..., p as one index varies from 1 to p while the other two indices remained fixed. Two Latin cubes of order p are equivalent if one can be obtained from the other by a permutation $\Pi = (\Pi_r, \Pi_c, \Pi_\ell)$, where Π_r is a permutation of the rows, Π_c is a permutation of the columns, and Π_ℓ is a permutation of the levels of C. Let G denote the group of all permutations so that G is isomorphic to S_p^3. We first prove

Lemma 3.1

Given three partitions of a prime p, each into at most $p - 1$ parts and not all into a single part, it is possible to select one part, say s_i, from each partition so that the least common multiple of two of the s_i's is less than $\operatorname{lcm}(s_1, s_2, s_3)$.

Proof: Number the partitions so that the first has more than one part and select as s_1 some part other than 1 from the first partition. Since p is prime, from the second partition we may select as s_2 some part such that $(s_2, s_1) = 1$. Similarly, select s_3 from the third partition so that $(s_3, s_1) = 1$ and, hence, $\mathrm{lcm}(s_2, s_3) < \mathrm{lcm}(s_1, s_2, s_3)$.

Corresponding to Lemma 1 of [3], we have

Lemma 3.2

Let $\Pi = (\Pi_r, \Pi_c, \Pi_\ell) \in G$. A Latin cube of order p a prime is nontrivially invariant under Π only if each component of Π is either a p-cycle or the identity and at least two of the components are p-cycles.

Proof: The permutation Π induces three partitions of p and if s_i is a part from the i^{th} partition for $i = 1, 2, 3$, we may assume that

$$\mathrm{lcm}(s_1, s_2) < \mathrm{lcm}(s_1, s_2, s_3).$$

If $\pi = (\Pi_1, \Pi_2, \Pi_3)$, let $(\ell_{i1}\ell_{i2}\ldots\ell_{is_i})$ be the corresponding cycle of the permutation Π_i. Tracing the effect of the cycles beginning with position $(\ell_{11}, \ell_{21}, \ell_{31})$ we get, after applying the permutation Π $d = \mathrm{lcm}(s_1, s_2)$ times that

$$(\ell_{11}, \ell_{21}, \ell_{31}) \to (\ell_{12}, \ell_{22}, \ell_{32}) \to \cdots \to (\ell_{11}, \ell_{21}, \ell_{3d}),$$

where $\ell_{3d} \neq \ell_{31}$ since $\mathrm{lcm}(s_1, s_2, s_3) > \mathrm{lcm}(s_1, s_2)$. For invariance, the elements in these positions must be equal, a contradiction of the Latin property. Hence all of the s_i must be 1 or p. If only one component contained a p cycle while the other two contained the identity, clearly the cube cannot be invariant without contradicting the Latin property.

Let $L(p, p, p)$ denote the number of Latin cubes of order p a prime. Clearly, if Π is the identity, then $L(p, p, p)$ cubes are invariant under Π, while there are $3[(p-1)!]^2$ permutations $\Pi = (\Pi_r, \Pi_c, \Pi_\ell)$ with the property that one of the components is the identity, while the other two consist of p-cycles. Moreover, each such permutation leaves $L(1, p, p) = L(p, p)$ cubes invariant. In order to count the number of cubes invariant under Π, where Π_r, Π_c, and Π_ℓ all consist of p-cycles, we need the following definitions and lemmas.

Definition 3.1

A *transversal* of a Latin square of order p is a set of p cells, one in each row and one in each column such that no two of the cells contain the same symbol.

Definition 3.2

A Latin square of order p is in *diagonal transversal* form if it consists of p disjoint transversals, one of which is the main diagonal and the remaining transversals are parallel to it, i.e., with addition mod p, cells (i, j) and $(i + 1, j + 1)$ are always in the same transversal.

Let d_p denote the number of Latin squares of order p in diagonal transversal form. We can now prove

Lemma 3.3

A Latin cube of order p a prime is invariant under a permutation $\Pi = (\Pi_r, \Pi_c, \Pi_\ell)$ where Π_r, Π_c, and Π_ℓ are p-cycles only if level one consists of p disjoint transversals.

Proof: Let $\Pi_r = (r_1 r_2 \ldots r_p)$ and $\Pi_c = (c_1 c_2 \ldots c_p)$. Consider the elements in the p positions $(r_1, c_1, \cdot)$ to be some permutations of 1, 2, ..., p. Repeated applications of Π carries these positions into the positions $(r_2, c_2, \cdot)$, $(r_3, c_3, \cdot)$, ... Since Π_ℓ is a p-cycle, each element occupies the position in level one in exactly one of the p sets of positions, and thus the elements in positions $(r_1, c_1, 1)$, $(r_2, c_2, 1)$, ..., $(r_p, c_p, 1)$ form a transversal. Similarly, successive applications of Π to the p positions $(r_2, c_1, \cdot)$ fixes $(r_3, c_2, \cdot)$, ..., $(r_1, c_p, \cdot)$ so that $(r_2, c_1, 1)$, ..., $(r_1, c_p, 1)$ is a second transversal in the first level. It thus follows that level one consists of p disjoint transversals.

Lemma 3.4

For p a prime there are d_p Latin cubes of order p invariant under a permutation $\Pi = (\Pi_r, \Pi_c, \Pi_\ell)$ where Π_r, Π_c, and Π_ℓ are p-cycles.

Proof: Suppose $\Pi_r = (1 i_2 \ldots i_p)$, $\Pi_c = (1 j_2 \ldots j_p)$, and $\Pi_\ell = (1 k_2 \ldots k_p)$. By the previous lemma, a cube will be invariant under Π only if level one consists of the disjoint transversals

$$\begin{array}{ll} T_1 & (1, 1), (i_2, j_2), \ldots, (i_p, j_p), \\ T_2 & (1, j_2), (i_2, j_3), \ldots, (i_p, 1), \\ \vdots & \qquad\qquad \vdots \\ T_p & (1, j_p), (i_2, 1), \ldots, (i_p, j_{p-1}). \end{array} \tag{3.1}$$

Rearrange the rows and columns by using the permutations

$$\begin{pmatrix} 1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & j_2 & \ldots & j_p \\ 1 & 2 & \ldots & p \end{pmatrix}$$

so that level one now consists of the transversals

$$\begin{array}{ll} T_1 & (1, 1), (2, 2), \ldots, (p, p), \\ T_2 & (1, 2), (2, 3), \ldots, (p, 1), \\ \vdots & \qquad\qquad \vdots \\ T_p & (1, p), (2, 1), \ldots, (p, p-1). \end{array}$$

Hence, level one is in diagonal transversal form so that the number of cubes invariant under Π is less than or equal to d_p.

Similarly, if we consider a Latin square of order p in diagonal transversal form and apply the permutations

$$\begin{pmatrix} 1 & 2 & \ldots & p \\ 1 & i_2 & \ldots & i_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & \ldots & p \\ 1 & j_2 & \ldots & j_p \end{pmatrix}$$

we obtain a square with p disjoint transversals as in (3.1). If we use this square as level one of a cube and allow $\Pi = (\Pi_r, \Pi_c, \Pi_\ell)$ to fix the remaining levels we will have constructed a cube invariant under Π so that d_p is no larger than the number of cubes invariant under Π.

It may be of interest to note that for $p = 2, 3$, and 5, $d_p = (p - 2)p!$. For p prime, one can construct a square in diagonal transversal form by choosing the first row in one of $p!$ ways and then rotating the row one position to the left $p - 1$ times to obtain the remaining rows. By making the $p - 1$ rotations each two positions to the left, one obtains a second diagonal transversal square with a given first row. Similarly, for left rotations of any fixed size up to and including $p - 2$ positions, a new diagonal transversal square is obtained so that $d_p \geqslant (p - 2)p!$. If $p = 7$, the following square

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 7 & 5 & 6 & 1 & 4 \\
7 & 5 & 4 & 6 & 1 & 2 & 3 \\
4 & 1 & 6 & 2 & 3 & 7 & 5 \\
3 & 6 & 5 & 1 & 7 & 4 & 2 \\
6 & 7 & 2 & 3 & 4 & 5 & 1 \\
5 & 4 & 1 & 7 & 2 & 3 & 6
\end{array}$$

is not obtained by a rotation of the first row so that $d_7 > 5 \cdot 7!$. Moreover, in general, if $p \geqslant 7$, we have $d_p > (p - 2)p!$. It would be of interest to have an exact formula for d_p for all p.

We now apply Burnside's lemma to prove

Theorem 3.1

Permutations of rows, columns, and levels induce

$$N_p = \frac{1}{(p!)^3}[L(p, p, p) + 3((p - 1)!)^2 L(p, p) + ((p - 1)!)^3 d_p]$$

equivalence classes in the set of Latin cubes of order p a prime.

If c_p is the number of reduced Latin cubes of order p, then

$$L(p, p, p) = p!(p - 1)!(p - 1)!c_p,$$

so that N_p may be written in the form

$$N_p = \frac{1}{p^3}[pc_p + 3p!\ell_p + d_p].$$

In [4] it was shown that $c_2 = c_3 = 1$ and $c_5 = 40{,}246$. Therefore, it is easily checked that $N_2 = N_3 = 1$, while $N_5 = 1774$.

4. HYPERCUBES

In this section we extend our results concerning squares and cubes of prime order to n-dimensional hypercubes of prime order. A Latin hypercube A of dimension n and order p is a $p \times p \times \cdots \times p$ array with the property that each of the p^n elements $a_{i_1 \ldots i_n}$ is one of the numbers $1, 2, \ldots, p$ and $\{a_{i_1 \ldots i_n}\}$ ranges over all of the numbers $1, 2, \ldots, p$ as one index varies from 1 to p, while the remaining indices are fixed. Let $L(n; p)$ be the number of n-dimensional Latin hypercubes of order p. We may generalize the proof of Lemma 3.1 to obtain

Lemma 4.1

Given n partitions of a prime p, each into at most $p - 1$ parts and not all into a single part, it is possible to select one part s_i from each partition so that the least common multiple of $n - 1$ of the s_i's is less than $\operatorname{lcm}(s_1, s_2, \ldots, s_n)$.

Let G be the group that permutes n-dimensional hypercubes by permuting each component so that G is isomorphic to S_p^n. Along the same lines as Lemma 3.2, we may prove

Lemma 4.2

Let $\Pi = (\Pi_1, \ldots, \Pi_n) \in G$. A Latin hypercube of order p a prime is nontrivially invariant under Π only if each Π_i is a p-cycle or the identity and at least two of the Π_i are p-cycles.

Definition 4.1

A *hypertransversal* of an n-dimensional Latin hypercube of order p is a collection of p cells $(i_1^k, \ldots, i_n^k)$, $k = 1, \ldots, p$, such that the corresponding p elements are distinct and among the p n-tuples, the set of p elements in each of the n coordinates is a permutation of $1, 2, \ldots, p$.

By extending the argument used in the proof of Lemma 3.3 to n dimensions, we may prove

Lemma 4.3

An n-dimensional Latin hypercube of order p a prime is invariant under a permutation $\Pi = (\Pi_1, \ldots, \Pi_n)$, where $\Pi_1, \ldots, \Pi_n$ are all p-cycles only if the hypercube possesses a subhypercube of dimension $n - 1$ that is composed of p^{n-2} disjoint hypertransversals.

Definition 4.2

An n-dimensional Latin hypercube of order p is in *parallel hypertransversal* form if it consists of p^{n-1} disjoint hypertransversals

$$(1, i_2, \ldots, i_n),\ (2, i_2 + 1, \ldots, i_n + 1), \ldots, (p, i_2 + p - 1, \ldots, i_n + p - 1),$$

where $(i_2, \ldots, i_n)$ ranges over all p^{n-1} $(n - 1)$-tuples and the additions are mod p.

Let $d(n; p)$ denote the number of n-dimensional Latin hypercubes in parallel hypertransversal form. Analogous to Lemma 3.4, we can prove

Lemma 4.4

For p a prime there are $d(n - 1; p)$ Latin n-dimensional hypercubes of order p invariant under a permutation $\Pi = (\Pi_1, \ldots, \Pi_n)$, where each Π_i is a p-cycle.

Theorem 4.1

Permutations of each coordinate induce

$$N_p = \frac{1}{(p!)^n}\left[L(n;\ p) + \sum_{k=2}^{n-1}\binom{n}{k}((p-1)!)^k L(k;\ p) + ((p-1)!)^n d(n-1;\ p)\right]$$

equivalence classes in the set of n-dimensional Latin hypercubes of order p a prime.

Proof: Clearly, $L(n;\ p)$ hypercubes are invariant under the identity and there are

$$\binom{n}{k}((p-1)!)^k$$

permutations $\Pi = (\Pi_1, \ldots, \Pi_n)$, where $n - k$ of the Π_i are the identity. Moreover, each of these fixes $L(k;\ p)$ k-dimensional hypercubes of order p. Applying Lemma 4.4 and Burnside's lemma yields the result.

REFERENCES

1. J. Dénes & A. D. Keedwell. *Latin Squares and Their Applications.* New York: Academic Press, 1974.
2. J. R. Hamilton & G. L. Mullen. "How Many $i - j$ Reduced Latin Rectangles Are There?" *Amer. Math. Monthly* 87 (1980);392-94.
3. C. Laywine. "An Expression for the Number of Equivalence Classes of Latin Squares Under Row and Column Permutations." *J. Comb. Thy.*, Series A, 30 (1981):317-20.
4. G. L. Mullen & R. E. Weber. "Latin Cubes of Order $\leqslant 5$." *Discrete Math.* 32 (1980):291-97.

◆◇◆◇◆