A NOTE ON A FIBONACCI IDENTITY

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The Fibonacci numbers are defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 2. \tag{1}$$

The following well-known identity relates F_n to the binomial coefficients:

$$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}. \tag{2}$$

In this note we give an interpretation of the individual terms in the identity (2) in terms of the original "rabbit problem":

Given a new born pair of rabbits on the first day of a month, find the number of pairs of rabbits at the end of n *months, assuming that each pair begets a pair each month starting when they are two months old.*

F_n is the number of pairs of rabbits at the end of n months. Now let

$S(n, k)$ = the number of pairs of k^{th} generation rabbits at the end of the n^{th} month.

Here the initial pair of rabbits is called the *zeroeth generation*, the immediate offspring of the initial pair are called *first generation* rabbits, the immediate offspring of the first-generation rabbits are called *second generation* rabbits, and so on.

We can now state our

Theorem

$$S(n, k) = \binom{n-k-1}{k}, \quad 0 \leqslant k \leqslant \left[\frac{n-1}{2}\right].$$

Proof: We have the simple accounting equation:

$$S(n, k) = S(n-1, k) + S(n-2, k-1). \tag{4}$$

This merely states that the number of k^{th} generation pairs at the end of the n^{th} month is equal to the number of such pairs at the end of the $(n-1)^{\text{st}}$ month plus the births of k^{th} generation rabbits during the n^{th} month. However, the births of k^{th} generation rabbits during the n^{th} month must come from $(k-1)^{\text{st}}$ generation rabbits who are at least two months old; there are precisely

$$S(n-2, k-1)$$

such pairs. Since there is only one zeroeth generation pair, we must have

$$S(n, 0) = 1. \tag{5}$$

To complete the proof, it is necessary only to verify that

$$S(n, k) = \binom{n-k-1}{k} \tag{6}$$

satisfies (4) and (5). Putting $k = 0$ in (6) we find $S(n, 0) = 1$. Substituting (6) into (4) we obtain

$$\binom{n - k - 1}{k} = \binom{n - k - 2}{k} + \binom{n - k - 2}{k - 1}$$

However, this is a well-known identity (see, e.g., [1, p. 70]).

For example, if we put $n = 12$ in identity (2), we find

$$F_{12} = 144 = 1 + 10 + 36 + 56 + 35 + 6.$$

Thus, among the 144 pairs of rabbits at the end of 12 months, there are, in addition to the initial pair, 10 first generation, 36 second generation, 56 third generation, 35 fourth generation, and 6 fifth generation pairs.

REFERENCE

1. J. G. Kemeny, J. L. Snell, & G. L. Thompson. *Introduction to Finite Mathematics*. 3rd ed. Englewood Cliffs, N.J.: Prentice-Hall, 1974.

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