HYPERPERFECT AND UNITARY HYPERPERFECT NUMBERS

WALTER E. BECK

University of Northern Iowa, Cedar Falls, IA 50613

RUDOLPH M. NAJAR

University of Wisconsin-Whitewater, Whitewater, WI 53190

(Submitted April 1984)

INTRODUCTION

In this paper, k, m, and n will represent arbitrary natural numbers; p, q, r, s, primes; and a, b, c, d, natural number exponents. σ is the sum-of-divisors function; σ^*, the sum-of-unitary divisors function; and τ, the count-of-prime-factors function.

Definition 1 [6]: A number m is said to be n-hyperperfect, n-HP, if it satisfies

$$m = 1 + n[\sigma(m) - m - 1]. \tag{1}$$

Definition 2 [2]: A number m is said to be n-unitary hyperperfect, n-UHP, if it satisfies

$$m = 1 + n[\sigma^*(m) - m - 1]. \tag{2}$$

For $n = 1$, the definitions reduce to those of the usual perfect and unitary perfect numbers. The two definitions agree on square-free numbers. To speak of both concepts simultaneously, we subsume equations (1) and (2) into

$$m = 1 + n[\Sigma(m) - m - 1] \tag{3}$$

and speak of n-(unitary) hyperperfect numbers, n-(U)HP.

1. PARITY

Theorem 1: Let m be n-(U)HP. Then:

(a) $(m, n) = 1$;
(b) If m is even, n is odd;
(c) If n is even, m is odd;
(d) $(m, \Sigma(m) - m - 1) = 1$;
(e) $(m, \Sigma(m) - 1) = 1$;
(f) $\tau(m) > 1$.

Proof: (a-e) Follow directly from (3).

(f) By contradiction. If $m = p^a$, $a > 1$, then

$$p \; (m, \Sigma(m) - 1)$$

which contradicts (e). ■

The possibility that both m and n are odd is not addressed in this theorem. The table of hyperperfect numbers in [3] includes odd m for odd n. For example, 325 is 3-HP. In the unitary case, we have a complete result.

Theorem 2: If m is n-UHP, then not both m and n are odd.

Proof: By contradiction. Assume $m = 2s + 1$, $n = 2t + 1$. Equation (2) becomes

$$2s + 1 = 1 + (2t + 1)[\sigma^*(m) - (2s + 1) - 1].$$

Expand and regroup.

$$4s + 2 = (2t + 1)\sigma^*(m) - 4ts - 4t.$$

Reduce modulo 4, remembering that $2t + 1$ is odd.

$$\sigma^*(m) \equiv 2 \bmod 4. \tag{4}$$

For (4) to be true, $\tau(m) = 1$. This contradicts Theorem 1(f). ■

Theorems 1 and 2 say that if m is n-UHP, not only are m and n relatively prime, they must be of opposite parity. The case in which $n = 1$ reduces to an old result.

Corollary 1 [7]: There are no odd unitary perfect numbers.

2. STRUCTURE THEOREMS

Equation (3) can also be written in the form

$$(n + 1)m = n\Sigma(m) - (n - 1). \tag{5}$$

Theorem 3: If m is n-HP, n odd, then m has as a component an odd prime to an odd power.

Proof: Let $m = 2^a m'$, $(2, m') = 1$. Equation (5) becomes

$$(n + 1)2^a m' = n\sigma(2^a)\sigma(m') - (n - 1).$$

The first and third terms are even since n is odd; n and $\sigma(2^a)$ are odd. Therefore $\sigma(m')$ is even. This happens only if an odd prime factor of m occurs to an odd power. ■

This argument yields no information in the unitary case, because $\sigma^*(m')$ is even. Note that the argument does not depend on a; it holds for $a = 0$.

Theorem 4: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$(p^a - n)(m' - n) \geqslant n^2 + 1.$$

Proof: Equation (5) becomes

$$(n + 1)m = n(p^a + 1)\sigma^*(m') - (n - 1) = np^a\sigma^*(m') + n\sigma^*(m') - (n - 1)$$

$$(n + 1)p^a m' - np^a\sigma^*(m') = n\sigma^*(m') - (n - 1)$$

$$p^a[(n + 1)m' - n\sigma^*(m')] = n\sigma^*(m') - (n - 1)$$

$$p^a = \frac{n\sigma^*(m') - (n - 1)}{(n + 1)m' - n\sigma^*(m')} \tag{6}$$

$\sigma^*(m') \geqslant m' + 1$ implies

$$(n + 1)m' - n\sigma^*(m') \leqslant (n + 1)m' - n(m' + 1); \tag{7}$$

and

$$\frac{n\sigma^*(m') - (n - 1)}{(n + 1)m' - n\sigma^*(m')} \geqslant \frac{n(m' + 1) = (n - 1)}{(n + 1)m' - n(m' + 1)} = \frac{nm' + 1}{m' - n} = n + \frac{n^2 + 1}{m' - n}.$$

Thus,

$$p^a \geqslant n + \frac{n^2 + 1}{m' - n}$$

or

$$(p^a - n)(m' - n) \geqslant n^2 + 1. \blacksquare$$

Corollary 2: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$\frac{n + 1}{n} > \frac{\sigma^*(m')}{m'}$$

Proof: In (6), the numerator is positive; hence, so is the denominator: $(n + 1)m' - n\sigma^*(m') > 0$.

The inequality follows immediately. $\blacksquare$

Corollary 3: Let m be n-UHP, $m = p^a m'$, $(p, m') = 1$. Then

$$\frac{n + 1}{n} > \frac{m' + 1}{m'}$$

Proof: $\sigma^*(m') \geqslant m' + 1$. Alternatively, the right side of (7) is positive, as the left side is. $\blacksquare$

Corollary 4: Let m be n-UHP, $m = p^a q^b$. Then

$$(p^a - n)(q^b - n) = n^2 + 1.$$

Proof: In Theorem 4, $m' = q^b$. $\sigma^*(q^b) = q^b + 1$. Equation (7) is an equality. The result follows. $\blacksquare$

Corollary 5: For given n, there are finitely many m of the form $m = p^a q^b$ which are n-UHP.

Proof: From Corollary 4,

$$p^a = n + \frac{n^2 + 1}{q^b - n} \quad \text{and} \quad q^b = n + \frac{n^2 + 1}{p^a - n}.$$

There are finitely many solutions for p^a, q^b. $\blacksquare$

Corollary 6: There is exactly one unitary perfect number with two distinct prime divosors.

Proof: In Corollary 5, $n = 1$, $n^2 + 1 = 2$. There is only one solution for p^a, q^b; namely, 2, 3. $m = 6$. $\blacksquare$

Corollary 7: Let m be n-UHP, $p^a \| m$. Then $p^a > n$.

Proof: This is the penultimate inequality in the proof of Theorem 4. ■

The import of Corollary 7 is that, if m is n-UHP, then all unitary divisors of m, except 1, exceed n. In the nonunitary case, every divisor of m, except 1, exceeds n ([6], Theorem 1). Minoli and Bear ([6], Theorem 3) demonstrate bounds on the prime factors of an n-HP number of the form $m = pq$. These bounds can be proved for the unitary case with some generalization.

Corollary 8: Let m be n-UHP, $m = p^a q^b$, $p^a < q^b$. Then:

(a) If $n > 1$, $n < p^a < 2n < q^b \leqslant n^2 + n + 1$;

(b) If $n = 1$, $n < p^a \leqslant 2n < q^b \leqslant n^2 + n + 1$.

Further,

(c) For $n = 1, 2$, there are unique solutions.

Proof: The first inequality is Corollary 7. The last inequality arises from Corollary 4.

$$n^2 + 1 = (p^a - n)(q^b - n) \geqslant q^b - n;$$

thus,

$$q^b \leqslant n^2 + n + 1.$$

For the second inequality, rewrite equation (2) as

$$p^a q^b = 1 + np^a + nq^b < 1 + 2nq^b$$

$$p^a q^b \leqslant 2nq^b$$

$$p^a \leqslant 2n.$$

If $p = 2$, by Theorem 1, n is odd. Thus, equality is possible only for $n = 1$, $p^a = 2$. Equation (2) also yields

$$p^a q^b \geqslant 2np^a$$

$$q^b \geqslant 2n.$$

Again, if $q = 2$, n is odd. Equality is possible only for $n = 1$, $q^b = 2$. Then $\tau(m) = 1$, which contradicts Theorem 1 and the initial assumption. This completes the proof of the inequalities. For $n = 1$, they reduce to

$$1 < p^a \leqslant 2 < q^b \leqslant 3.$$

The only solution is $p^a = 2$; $q^b = 3$, $m = 6$. For $n = 2$,

$$2 < p^a < 4 < q^b \leqslant 7;$$

thus, $p^a = 3$. By Corollary 4, $q^b = 7$. ■

Theorem 5: If m is n-(U)HP, then

$$\frac{n}{n+1} \geqslant \frac{m}{\Sigma(m)} > \left(\frac{n}{n+1}\right)\left(\frac{m-1}{m}\right),$$

with equality on the left if and only if $n = 1$.

Proof: On division by $(n + 1)\Sigma(m)$, equation (5) becomes

$$\frac{m}{\Sigma(m)} = \frac{n}{n+1} - \frac{n-1}{(n+1)\Sigma(m)}. \tag{8}$$

The left inequality is immediate.

As $\Sigma(m) > m$,

$$\frac{n-1}{(n+1)\Sigma(m)} \leqslant \frac{n-1}{(n+1)m} \quad \text{and} \quad -\frac{n-1}{(n+1)\Sigma(m)} \geqslant -\frac{n-1}{(n+1)m}.$$

Equation (8) yields

$$\frac{m}{\Sigma(m)} \geqslant \frac{n}{n+1} - \frac{n-1}{(n+1)m} = \frac{nm-n+1}{(n+1)m} > \frac{nm-n}{(n+1)m} = \left(\frac{n}{n+1}\right)\left(\frac{m-1}{m}\right),$$

which is the inequality on the right. ■

Results on mod 3 properties have appeared before. In particular, Hagis [2] proved the following.

Theorem 6: Let m be n-UHP, then:

(a) If $m \not\equiv 0 \bmod 3$, then $m \equiv 1 \bmod 3$;
(b) If $n \equiv 0 \bmod 3$, then $m \equiv 1 \bmod 3$;
(c) If $n \equiv 1 \bmod 3$, then $\sigma^*(m) \equiv 2m \bmod 3$;
(d) If $n \equiv -1 \bmod 3$, then $\sigma^*(m) \equiv 2 \bmod 3$.

Results (b), (c), and (d) follow immediately from equation (3) and so are valid for the (ordinary) hyperperfect case also.

3. UNITARY HYPERPERFECT NUMBERS

The set of unitary hyperperfect numbers has nonempty intersections with the set of (ordinary) hyperperfect numbers and with the set of unitary perfects. In the first case, the intersection is the set of square-free hyperperfect numbers. In the second, it is the set (see [7], [11]) of 1-unitary hyperperfect numbers. For square-free hyperperfect numbers, see [4], [5], [6], [8], [9], and [10].

Hagis [2] ran a computer search for unitary hyperperfect numbers through 10^6. Buell [1] found 146 unitary hyperperfect numbers less than 10^8.

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