ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.: Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-550 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Show that the powers of -13 form a Fibonacci-like sequence modulo 181, that is, show that

$$(-13)^{n+1} \equiv (-13)^n + (-13)^{n-1} \pmod{181} \text{ for } n = 1, 2, 3, \ldots .$$

B-551 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Generalize on Problem B-550.

B-552 *Proposed by Philip L. Mana, Albuquerque, NM*

Let S be the set of integers n with $10^9 < n < 10^{10}$ and with each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appearing (exactly once) in n.

(a) What is the smallest integer n in S with $11|n$?
(b) What is the probability that $11|n$ for a randomly chosen n in S?

B-553 *Proposed by D. L. Muench, St. John Fisher College, Rochester, NY*

Find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} L_{i+1}^2$.

B-554 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in $Z^+ = \{1, 2, \ldots\}$, prove that there exist x and y in Z^+ such that

$$(F_{4n-1} + 1)(F_{4n+1} + 1) = x^2 + y^2.$$

B-555 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

For all n in Z^+, prove that there exist x, y, and z in Z^+ such that

$$(F_{2n-1} + 4)(F_{2n+5} + 1) = x^2 + y^2 + z^2.$$

SOLUTIONS

Quadratic with an Integer Solution

B-526 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Find all ordered pairs (m, n) of positive integers for which there is an integer x satisfying the equation

$$F_mF_nx^2 - [F_m(F_m, F_n) + F_nF_{(m,n)}]x + (F_m, F_n)F_{(m,n)} = 0.$$

Here (r, s) denotes the greatest common divisor of r and s.

Solution by Paul S. Bruckman, Fair Oaks, CA

We use the well-known relation

$$(F_m, F_n) = F_{(m,n)}. \tag{1}$$

Then, letting $d = F_{(m,n)}$, the given equation becomes

$$(F_mx - d)(F_nx - d) = 0, \tag{2}$$

to be satisfied for some integer x. Since $m \geqslant (m, n)$, $n \geqslant (m, n)$ and $(F_n)_{n=1}^{\infty}$ is an increasing sequence (except for $F_1 = F_2 = 1$), we see that for $x = d/F_m$ to be an integer, we must have one of the following:

(a) $F_m = F_{(m,n)}$ or (b) $F_n = F_{(m,n)}$.

These, in turn, imply at least one of the following:

(i) $m = 1$; (ii) $m = 2$; (iii) $m|n$; (iv) $n = 1$; (v) $n = 2$; (vi) $n|m$.

Some of these cases are redundant, and we can consolidate them as follows: all ordered pairs $\{m, n\}$ with (a) $m|n$; (b) $n|m$; (c) $m = 2$; (d) $n = 2$. (Note that there is still some redundancy, but this is minimal.)

Also solved by Paul S. Bruckman, Laszlo Cseh, A. Di Porto & P. Filipponi, Herta T. Freitag, Walther Janous, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.

Another Quadratic with an Integer Solution

B-527 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Do as in B-526 with the equation replaced by

$$(F_m, F_n)x^2 - (F_m + F_n)x + F_{(m,n)} = 0.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

The given equation reads

$$F_{(m,n)}x^2 - (F_m + F_n)x + F_{(m,n)} = 0. \tag{1}$$

Since $(m, n)|m$ and $(m, n)|n$, it holds that $s_{m,n} = (F_m + F_n)/F_{(m,n)}$ is integral; that is, (1) reads $x^2 - s_{m,n}x + 1 = 0$. This symmetric equation has to have the double root $x_1 = x_2 = 1$, whence $F_m + F_n = 2F_{(m,n)}$.

Because $F_{(m,n)} \leqslant F_m$ and $F_{(m,n)} \leqslant F_n$, it follows that $F_m = F_n = F_{(m,n)}$. Thus, $m = n$ or $m = 1$, $n = 2$ or $m = 2$, $n = 1$.

Also solved by Paul S. Bruckman, A. Di Porto & P. Filipponi, Herta T. Freitag, L. Kuipers, Bob Prielipp, Sahib Singh, and the proposer.

Special Case of a Sum

B-528 *Proposed by Herta T. Freitag, Roanoke, VA*

For nonnegative integers n, prove that

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}F_{i+1}^2 = 5^nF_{2n+3}.$$

Solution by Marjorie Bicknell-Johnson, Santa Clara, CA

Let $p = 1$ in equation (4) on page 30 of the following article: "Some New Fibonacci Identities" by Verner E. Hoggatt, Jr., and Marjorie Bicknell, in *The Fibonacci Quarterly* 2, no. 1 (February 1964):29-32.

Also solved by Wray G. Brady, Paul S. Bruckman, Laszlo Cseh, Leonard A. G. Dresel, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, George N. Philippou, Bob Prielipp, A. G. Shannon, Sahib Singh, J. Suck, Robert L. Vogel, and the proposer.

Compact Form for a Sum

B-529 *Proposed by Herta T. Freitag, Roanoke, VA*

For positive integers n, find a compact form for

$$\sum_{i=0}^{2n}\binom{2n}{i}F_{i+1}^2.$$

Solution by Leonard A. G. Dresel, University of Reading, England

Let $T = \sum_{i=0}^{2n}\binom{2n}{i}F_{i+1}^2$. Then

$$5T = \sum\binom{2n}{i}(\alpha^{i+1} - \beta^{i+1})^2 = \sum\binom{2n}{i}(\alpha^{2i+2} - 2\alpha\beta(\alpha\beta)^i + \beta^{2i+2})$$

$$= \alpha^2(1 + \alpha^2)^{2n} - 2\alpha\beta(1 + \alpha\beta)^{2n} + \beta^2(1 + \beta^2)^{2n}.$$

Now, since $n > 0$ and $\alpha\beta = -1$, the middle term vanishes, and

$$\alpha^2 + 1 = \alpha(\alpha - \beta) = \sqrt{5}\alpha \quad \text{and} \quad \beta^2 + 1 = \beta(\beta - \alpha) = -\sqrt{5}\beta.$$

Hence,

$$T = 5^{n-1}(\alpha^{2n+2} + \beta^{2n+2}) = 5^{n-1}L_{2n+2}.$$

Also solved by Marjorie Bicknell-Johnson, Wray G. Brady, Paul S. Bruckman, Laszlo Cseh, Piero Filipponi, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, D. L. Muench, George N. Philippou, Bob Prielipp, A. G. Shannon, Sahib Singh, J. Suck, Robert L. Vogel, and the proposer.

Lucas Continued Fraction

B-530 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $\alpha = (1 + \sqrt{5})/2$. For n an odd positive integer, prove that the continued fraction

$$L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \cdots}} = \alpha^n.$$

Solution by Graham Lord, Princeton, NJ

The simple continued fraction is convergent (see Hardy & Wright, for example). The limit x satisfies the inequality $L_n \leqslant x$, and is a root of the equation $L_n + 1/x = x$. Since n is odd, the latter equation can be rewritten as

$$(x - \alpha^n)(x - \beta^n) = 0,$$

from which, together with the inequality, it follows that α^n is the required value.

Also solved by Wray Brady, Paul S. Bruckman, Laszlo Cseh, Walther Janous, A. Di Porto & P. Filipponi, Leonard A. G. Dresel, Herta T. Freitag, C. Georghiou, L. Kuipers, I. Merenyi, D. L. Muench, Bob Prielipp, Sahib Singh, Robert L. Vogel, and the proposer.

Even Case of Lucas Continued Fraction

B-531 *Proposed by Michael Eisenstein, San Antonio, TX*

For n an even positive integer, prove that

$$L_n - \cfrac{1}{L_n - \cfrac{1}{L_n - \cdots}} = \alpha^n.$$

Solution by Graham Lord, Princeton, NJ

The existence of the infinite continued fraction is first established. If x_k is the k^{th} convergent, then easy induction arguments show that

$$L_n - 1 \leqslant x_k \leqslant L_n,$$

and that

$$x_k + \frac{1}{x_k} > L_n;$$

the latter requires use of the identity

$$x_{k+1} + \frac{1}{x_{k+1}} = L_n + \left(x_k + \frac{1}{x_k} - L_n\right)\Big/(x_k L_n - 1).$$

So

$$x_k - x_{k+1} = x_k + \frac{1}{x_k} - L_n > 0.$$

Hence, x is a strictly decreasing sequence which is bounded below by $L_n - 1$; thus, the limit exists.

The value of the limit is a root of the equation $x = L_n - 1/x$, which can be rewritten as $(x - \alpha^n)(x - \beta^n) = 0$, since n is even. Because $x_k > L_n - 1$, the value of the continued fraction is α^n.

Also solved by Paul S. Bruckman, Laszlo Cseh, A. Di Porto & P. Filipponi, Leonard A. G. Dresel, Herta T. Freitag, C. Georghiou, Walther Janous, L. Kuipers, I. Merenyi, D. L. Meunch, Bob Prielipp, Sahib Singh, Robert L. Vogel, and the proposer.

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