ZIGZAG POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to obtain some basic properties of certain polynomials which we choose to call *zigzag* polynomials. These arise in a specified way from the diagonal terms of the Pascal-type array of polynomials generated by a given second-order recurrence relation.

Consider the sequence of *generalized Pell polynomials* $\{A_n(x)\}$ defined by

$$A_n(x) = 2xA_{n-1}(x) + A_{n-2}(x),\ A_0(x) = q,\ A_1(x) = p \quad (n \geqslant 2). \tag{1.1}$$

Special cases of $A_n(x)$ which will concern us are:

the *Pell polynomials* $P_n(x)$ occurring when $p = 1$, $q = 0$, (1.2)

the *Pell-Lucas polynomials* $Q_n(x)$ occurring when $p = 2x$, $q = 2$. (1.3)

The explicit *Binet form* for $A_n(x)$ is given in [4], namely,

$$A_n(x) = \frac{(p - q\beta)\alpha^n - (p - q\alpha)\beta^n}{\alpha - \beta}, \tag{1.4}$$

where α, β are the roots of $y^2 - 2xy - 1 = 0$ ($\alpha = x + \sqrt{x^2 + 1}$, $\beta = x - \sqrt{x^2 + 1}$). From (1.4), the Binet forms of $P_n(x)$ and $Q_n(x)$ are readily derived using (1.2) and (1.3).

The generating function for $\{A_n(x)\}$ is

$$\sum_{n=0}^{\infty} A_{n+1}(x)t^n = (p + qt)[1 - (2xt + t^2)]^{-1}. \tag{1.5}$$

Generating functions for $P_n(x)$ and $Q_n(x)$ are then, from (1.2), (1.3), and (1.5),

$$\sum_{n=0}^{\infty} P_{n+1}(x)t^n = [1 - (2xt + t^2)]^{-1} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} Q_{n+1}(x)t^n = (2x + 2t)[1 - (2xt + t^2)]^{-1}, \tag{1.7}$$

as given in [3].

Results (1.4)-(1.7) will not be used in this paper. Nevertheless, we append them here for reasons of completeness and comparison.

Though it will not interest us for the purpose of this paper, the curious reader may wish to investigate the special, simple case of (1.1) arising from the values $p = 1$, $q = 1$.

Background information for the theory about to be developed is to be found in [1] and [2].

2. ZIGZAG RISING DIAGONAL POLYNOMIALS

From (1.1), we form the Pascal-type array (Table 1).

$$\left\{\begin{array}{lllllllll}
 & & & & & & & & Z_0(x) \\
A_0(x) = q & & & & & & & & Z_1(x)\quad Z_2(x) \\
A_1(x) = p & & & & & & & & Z_3(x) \\
A_2(x) = 2px & + q & & & & & & & Z_4(x)\quad Z_5(x) \\
A_3(x) = 4px^2 & + 2qx & + p & & & & & & Z_6(x) \\
A_4(x) = 8px^3 & + 4qx^2 & + 4px & + q & & & & & Z_7(x)\quad Z_8(x) \\
A_5(x) = 16px^4 & + 8qx^3 & + 12px^2 & + 4qx & + p & & & & Z_9(x) \\
A_6(x) = 32px^5 & + 16qx^4 & + 32px^3 & + 12qx^2 & + 6px & + q & & & Z_{10}(x) \\
A_7(x) = 64px^6 & + 32qx^5 & + 80px^4 & + 32qx^3 & + 24px^2 & + 6qx & + p & & Z_{11}(x)\quad Z_{12}(x) \\
A_8(x) = 128px^7 & + 64qx^6 & + 192px^5 & + 80qx^4 & + 80px^3 & + 24qx^2 & + 8px & + q & \\
A_9(x) = 256px^8 & + 128qx^7 & + 448px^6 & + 192qx^5 & + 240px^4 & + 80qx^3 & + 40px^2 & + 8qx & + p \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots &
\end{array}\right. \tag{2.1}$$

Table 1. Zigzag Rising Diagonal Polynomials of $\{A_n(x)\}$

Let us agree to call the polynomials in Table 1 that arise upward in step-like formation from the left (indicated by lines) the *zigzag polynomials* (or *echelon polynomials*) associated with $\{A_n(x)\}$. At each level in the step-like formation, other than the first, the terms are paired in the second and third columns, the fourth and fifth columns, ..., where this is appropriate.

As will be evident in the next section, the value of this pairing technique is that specializations can be quickly visualized and obtained from the general pattern, e.g., by the disappearance of the first column of a pair when $p = 1$, $q = 0$ (the Pell polynomials), and by the amalgamation of corresponding elements in a pair of columns when $p = 2x$, $q = 2$, i.e., $p = qx$ (the Pell-Lucas polynomials).

Designate the zigzag polynomials by $Z_n(x)$. Start with $Z_0(x) = q$. Then, the first few zigzag polynomials are, from (2.1):

$$\begin{cases} Z_0(x) = q,\ Z_1(x) = p,\ Z_2(x) = 2px,\ Z_3(x) = 4px^2 + q, \\ Z_4(x) = 8px^3 + 2qx + p,\ Z_5(x) = 16px^4 + 4qx^2 + 4px, \\ Z_6(x) = 32px^5 + 8qx^3 + 12px^2 + q,\ Z_7(x) = 64px^6 + 16qx^4 + 32px^3 + 4qx + p, \\ Z_8(x) = 128px^7 + 32qx^5 + 80px^4 + 12qx^2 + 6px,\ \ldots \end{cases} \tag{2.2}$$

Using (1.1) and the nature of the formation of the $Z_n(x)$, we observe that

$$Z_n(x) = 2xZ_{n-1}(x) + Z_{n-3}(x). \tag{2.3}$$

Elementary methods applied to (2.3) produce the generating function for $Z_n(x)$, namely (when $n > 0$),

$$\sum_{n=1}^{\infty} Z_n(x)t^{n-1} = (p + qt^2)[1 - (2xt + t^3)]^{-1} \equiv Z(x, t). \tag{2.4}$$

Explicit formulation of an expression for $Z\ (x)$ can be obtained by comparison of coefficients of t in (2.4). Computation yields

$$Z_n(x) = p\sum_{i=0}^{\left[\frac{n-1}{3}\right]}\binom{n-1-2i}{i}(2x)^{n-1-3i} + q\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-3-2i}{i}(2x)^{n-3-3i}, \quad n \geqslant 3, \tag{2.5}$$

where $[n/3]$ is the integral part of $n/3$.

Certain differential equations are satisfied by the zigzag polynomials. These include the partial differential equation

$$2t\frac{\partial}{\partial t}Z(x,\ t) - (2x + 3t^2)\frac{\partial}{\partial x}Z(x,\ t) = 4qt^2[1 - (2xt + t^3)]^{-1} \tag{2.6}$$

and the ordinary differential equation

$$2x\frac{d}{dx}Z_{n+2}(x) + 3\frac{d}{dx}Z_n(x) = 2(n+1)Z_{n+2}(x) - 4qR_n(x), \tag{2.7}$$

where $R_n(x)$ is to be defined in the next section.

In deriving the results (2.5), (2.6), and (2.7), we have been guided by similar specialized results established in [2] for the rising diagonal polynomials $R_n(x)$ and $r_n(x)$. To these polynomials we now turn our attention.

3. SPECIALIZATIONS

Using (1.1), (1.2), and (1.3), we form Tables 2 and 3 for the polynomial sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$:

$$\begin{array}{lll} & & R_0(x) \\ \left\{\begin{array}{l} P_0(x) = 0 \\ P_1(x) = 1 \\ P_2(x) = 2x \\ P_3(x) = 4x^2 + 1 \\ P_4(x) = 8x^3 + 4x \\ P_5(x) = 16x^4 + 12x^2 + 1 \\ P_6(x) = 32x^5 + 32x^3 + 6x \\ P_7(x) = 64x^6 + 80x^4 + 24x^2 + 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. & & \begin{array}{l} R_1(x) \\ R_2(x) \\ R_3(x) \\ R_4(x) \\ R_5(x) \\ R_6(x) \\ R_7(x) \\ R_8(x) \end{array} \end{array} \tag{3.1}$$

Table 2. Rising Diagonal Polynomials of $\{P_n(x)\}$

Tables 2 and 3, it may be noted, are special cases of arrays given in [2]. Allowing for the necessary change of notation from [2] to this paper, denote the rising diagonal polynomials in Tables 2 and 3 by $R_n(x)$ and $r_n(x)$, respectively, commencing with $R_0(x) = 0$, $r_0(x) = 2$.

$$\left\{\begin{array}{lllllll}
 & & r_0(x) & & & & \\
Q_0(x) & = & 2 & r_1(x) & & & \\
Q_1(x) & = & 2x & r_2(x) & & & \\
 & & & & r_3(x) & & \\
Q_2(x) & = & 4x^2 & +\ 2 & r_4(x) & & \\
Q_3(x) & = & 8x^3 & +\ 6x & r_5(x) & & \\
 & & & & & r_6(x) & \\
Q_4(x) & = & 16x^4 & +\ 16x^2 & +\ 2 & r_7(x) & \\
Q_5(x) & = & 32x^5 & +\ 40x^3 & +\ 10x & & \\
Q_6(x) & = & 64x^6 & +\ 96x^4 & +\ 36x^2 & +\ 2 & \\
Q_7(x) & = & 128x^7 & +\ 224x^5 & +\ 112x^3 & +\ 14x & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}\right. \tag{3.2}$$

Table 3. Rising Diagonal Polynomials of $\{Q_n(x)\}$

Observe the relationships (cf. [2]), subject to the restriction $n \geqslant 3$,

$$\left\{\begin{array}{lll}
R_n(x) & = & 2xR_{n-1}(x) + R_{n-3}(x) \\
r_n(x) & = & 2xr_{n-1}(x) + r_{n-3}(x) \\
r_{n-1}(x) & = & R_n(x) + R_{n-3}(x).
\end{array}\right. \tag{3.3}$$

The formal structural equivalence of (2.3) and the first two equations in (3.3) is, of course, expected and essential.

Substituting the appropriate values from (1.2) and (1.3) in (2.5), we derive the explicit forms

$$R_n(x) = \sum_{i=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2i}{i} (2x)^{n-1-3i}, \quad n \geqslant 1, \tag{3.4}$$

and

$$r_n(x) = \sum_{i=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2i}{i} (2x)^{n-3i} + 2\sum_{i=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2i}{i} (2x)^{n-3-3i}, \quad n \geqslant 3. \tag{3.5}$$

Generating functions are, from (1.2), (1.3), and (2.4), when $n > 0$,

$$\sum_{n=1}^{\infty} R_n(x)t^{n-1} = [1 - (2xt + t^3)]^{-1} \equiv R(x, t) \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} r_n(x)t^{n-1} = 2(x + t^2)[1 - (2xt + t^3)]^{-1} \equiv r(x, t). \tag{3.7}$$

Furthermore, on applying (1.2) to (2.6) and (2.7) in succession, we deduce that

$$2t\,\frac{\partial R}{\partial t}(x, t) - (2x + 3t^2)\,\frac{\partial R}{\partial x}(x, t) = 0$$

and

$$2x \frac{d}{dx} R_{n+2}(x) + 3 \frac{d}{dx} R_n(x) = 2(n+1)R_{n+2}(x).$$

But we cannot apply (1.3) to (2.6) and (2.7) because, in (2.6) and (2.7), p and q were implicitly assumed to be constants, whereas in (1.3), $p = 2x$ and $q = 2$, i.e., p is a function of x.

Guided by the appropriate results in [2] and carrying out the processes of differentiation, *mutatis mutandis*, we arrive at the differential equations

$$2t \frac{\partial}{\partial t} r(x, t) - (2x + 3t^2) \frac{\partial}{\partial x} r(x, t) = r(x, t) - 6xR(x, t) \tag{3.10}$$

and

$$2x \frac{d}{dx} r_{n+2}(x) + 3 \frac{d}{dx} r_n(x) = 2(n-1)r_{n+2}(x) + 6R_{n+3}(x), \tag{3.11}$$

which should be compared with the corresponding results in [2].

Equations (3.3)-(3.9) occur in [2], slightly modified where necessary to take into account the minor differences in notation in [2] and in this paper.

In passing, it might be observed that a marginally neater form of (3.7) exists if the summation is allowed to commence with $n = 2$, instead of with $n = 1$ in conformity with (2.4). [Had our summation in (2.4) begun with $n = 0$, we would have obtained a slightly less simple form of the generating function than that given in (2.4).]

While there may be other mathematically interesting instances of $\{A_n(x)\}$, we have limited our attention to the two well-known and related sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$. Properties of $\{A_n(x)\}$ are an amalgam of their separate properties.

4. ORDINARY (NON-ZIGZAG) RISING DIAGONAL POLYNOMIALS

Consider next the ordinary (non-zigzag) rising diagonal polynomials in Table 1, which must not be confused with the $Z_n(x)$.

Denote these non-zigzag polynomials by the suggestive notation $\text{Ƶ}_n(x)$, beginning with $\text{Ƶ}_0(x) = q$.

Some of these polynomials are:

$$\begin{cases} \text{Ƶ}_0(x) = q,\ \text{Ƶ}_1(x) = p,\ \text{Ƶ}_2(x) = 2px,\ \text{Ƶ}_3(x) = 4px^2 + q, \\ \text{Ƶ}_4(x) = 8px^3 + 2qx,\ \text{Ƶ}_5(x) = 16px^4 + 4qx^2 + p, \\ \text{Ƶ}_6(x) = 32px^5 + 8qx^3 + 4px,\ \text{Ƶ}_7(x) = 64px^6 + 16qx^4 + 12px^2 + q, \\ \text{Ƶ}_8(x) = 128px^7 + 32qx^5 + 32px^3 + 4qx,\ \ldots \end{cases} \tag{4.1}$$

Observe that the recurrence relation for $\{\text{Ƶ}_n(x)\}$ is

$$\text{Ƶ}_n(x) = 2x\text{Ƶ}_{n-1}(x) + \text{Ƶ}_{n-4}(x). \tag{4.2}$$

Using elementary procedures, we may demonstrate that the (somewhat ungainly) generating function for $\text{Ƶ}_n(x)$ is

$$\sum_{n=0}^{\infty} \text{Ƶ}_n(x)t^n = \{q + (p - 2qx)t + qt^3\}[1 - (2xt + t^4)]^{-1}. \tag{4.3}$$

An explicit expression for the elements of $\{\text{Ƶ}\ (x)\}$ may be established, namely,

$$\mathcal{Z}_n(x) = p\sum_{i=0}^{\left[\frac{n-3}{3}\right]}\binom{n-1-3i}{i}(2x)^{n-1-4i} + q\sum_{i=0}^{\left[\frac{n-5}{3}\right]}\binom{n-3-3i}{i}(2x)^{n-3-4i}, \quad n \geqslant 5. \tag{4.4}$$

Finally, we emphasize that the rising diagonals $R_n(x)$ and $r_n(x)$ for $\{P_n(x)\}$ and $\{Q_n(x)\}$ in (3.1) and (3.2) are special cases of $Z_n(x)$, not $\mathcal{Z}_n(x)$, as a little thought reveals.

5. ZIGZAG DESCENDING DIAGONAL POLYNOMIALS

Just as the rising zigzag diagonal polynomials are constructed from Table 1, so the corresponding zigzag polynomials for descending diagonals may be generated, i.e., by proceeding downward in step-like fashion from the left.

To avoid repetitious waste of space, we invite the reader to refer to Table 1 and to compose the following list of descending diagonal *zigzag polynomials* (or echelon polynomials) $z_n(x)$, with initial value $z_0(x) = q$:

$$\begin{cases} z_0(x) = q,\ z_1(x) = p + q,\ z_2(x) = (p + q)(2x + 1), \\ z_3(x) = (p + q)(2x + 1)^2,\ z_4(x) = (p + q)(2x + 1)^3, \\ z_5(x) = (p + q)(2x + 1)^4,\ z_6(x) = (p + q)(2x + 1)^5,\ \ldots \end{cases} \tag{5.1}$$

The pattern is crystal clear. One does not have to be psychic to deduce immediately the recurrence relation from the geometric progression, namely,

$$z_{n+1}(x) = (2x + 1)z_n(x),\ n \geqslant 1, \tag{5.2}$$

with general term

$$z_n(x) = (p + q)(2x + 1)^{n-1},\ n \geqslant 1. \tag{5.3}$$

The generating function for $z_n(x)$ (if $n > 0$) is obviously

$$z(x, t) \equiv \sum_{n=1}^{\infty} z_n(x)t^{n-1} = (p + q)[1 - (2x + 1)t]^{-1}. \tag{5.4}$$

Mathematical calculations involving $z_n(x)$ will be manifestly simpler than those associated with $Z_n(x)$. In particular, the following differential equations flow easily from (5.3) and (5.4):

$$2t\frac{\partial}{\partial t}z(x, t) - (2x + 1)\frac{\partial}{\partial x}z(x, t) = 0 \tag{5.5}$$

$$(2x + 1)\frac{d}{dx}z_n(x) - 2(n - 1)z_n(x) = 0. \tag{5.6}$$

Specializations of (5.3)-(5.6) for $\{P_n(x)\}$ and $\{Q_n(x)\}$ are readily obtained. Thus, for the descending diagonal polynomials $D_n(x)$ of the Pell polynomial array in Table 2, with initial conditions $D_0(x) = 0$ and $D_1(x) = 1$, we derive

$$D_n(x) = (2x + 1)^{n-1},\ n \geqslant 1, \tag{5.7}$$

$$D(x, t) \equiv \sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1 - (2x + 1)t]^{-1}, \tag{5.8}$$

$$2t\frac{\partial}{\partial t}D(x, t) - (2x + 1)\frac{\partial}{\partial x}D(x, t) = 0, \tag{5.9}$$

$$(2x + 1)\frac{d}{dx}D_n(x) - 2(n - 1)D_n(x) = 0, \tag{5.10}$$

while, for the descending diagonal polynomials $d_n(x)$ of the Pell-Lucas polynomial array in Table 3, we deduce

$$d_n(x) = 2(x + 1)(2x + 1)^{n-1}, \quad n \geqslant 1, \tag{5.11}$$

$$d(x, t) \equiv \sum_{n=1}^{\infty} d_n(x)t^{n-1} = 2(x + 1)[1 - (2x + 1)t]^{-1}. \tag{5.12}$$

Initially, $d_0(x) = 2$.
Observe that

$$d_n(x) = D_n(x) + D_{n+1}(x). \tag{5.13}$$

Equations (5.5) and (5.6) cannot be applied directly to $d_n(x)$ since, in this case, $p = 2x$ is not a constant (although $q = 2$ is). However, the results for $d(x, t)$ and $d_n(x)$ corresponding to those for $D(x, t)$ and $D_n(x)$ in (5.9) and (5.10), respectively, may be established without too much difficulty if we permit ourselves to be assisted by similar results in [2]. They are:

$$2t \frac{\partial}{\partial t} d(x, t) - (2x + 1)\left[\frac{\partial}{\partial x} d(x, t) - 2D(x, t)\right] = 0 \tag{5.14}$$

$$2(x + 1) \frac{d}{dx}(d_{n+1}(x)) - 2d_{n+1}(x) - 8n(x + 1)^2 D_n(x) = 0. \tag{5.15}$$

The above specializations should be compared with analogous derivations in [2], modified as demanded by the circumstances. Variations that occur between a result in [2] and a corresponding result in this paper exist because of the different starting points, i.e., different values of $d_1(x)$.

Earlier results obtained in [1] relating to material in this paper might also be consulted.

6. CONCLUDING COMMENTS

This completes what we wished to say about the zigzag polynomials at this stage. Various generalizations of aspects of this paper suggest themselves, but, as we believe these developments go beyond the unity of this paper, they are left for possible further consideration.

Finally, it might be observed that results (2.3), (3.3), (4.2), (5.2), (5.7) and (5.11) are readily established by using the rule of formation and the generating functions for the columns of the respective arrays. In Table 1, for instance, the generating functions for the first, second, third, ..., pair of columns are $(1 - 2x)^{-1}$, $(1 - 2x)^{-2}$, $(1 - 2x)^{-3}$, ..., with appropriate multipliers p and q.

REFERENCES

1. A. F. Horadam. "Diagonal Functions." *The Fibonacci Quarterly 16*, no. 1 (1978):33-36.
2. A. F. Horadam. "Extensions of a Paper on Diagonal Functions." *The Fibonacci Quarterly 18*, no. 1 (1980):3-8.
3. A. F. Horadam & J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly 23*, no. 1 (1985):7-20.
4. J. E. Walton & A. F. Horadam. "Generalized Pell Polynomials and Other Polynomials." *The Fibonacci Quarterly 22*, no. 4 (1984):336-339.

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