ON A FIBONACCI ARITHMETICAL TRICK

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1. INTRODUCTION

A standard arithmetical trick for school children is to ask them to choose two positive integers, to extend this to a sequence of 10 numbers by adding any two to obtain the next in the Fibonacci manner, and then to add up the numbers in the sequence. When the exercise is complete the teacher, having unobtrusively noted the seventh number in each student's sequence while checking around the room to see that each is proceeding properly, can mystify the students by announcing the sum each has achieved. Given that the students did the arithmetic correctly, the sum is just 11 times the seventh number in their original sequence. If, for example, a student chooses 5 and 1, his sequence is

$$5,\ 1,\ 6,\ 7,\ 13,\ 20,\ 33,\ 53,\ 86,\ 139$$

and the sum is $363 = 11 \cdot 33$.

Of course, as the reader will expect, this is just a special case of more general results which we now examine.

2. SOME GENERAL RESULTS

Let F_n and L_n denote, respectively, the n^{th} Fibonacci and Lucas numbers so that

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \text{ for } n \geqslant 1,$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1} \text{ for } n \geqslant 1.$$

Also, define sequences H_n and K_n for integers a and b by

$$H_1 = a,\ H_2 = b,\ H_{n+2} = H_{n+1} + H_n \text{ for } n \geqslant 1,$$

and

$$K_1 = -a + 2b,\ K_2 = 2a + b,\ K_{n+2} = K_{n+1} + K_n \text{ for } n \geqslant 1.$$

Then the following theorem holds.

Theorem 1: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} H_i = L_{2n-1}H_{2n+1}, \qquad \sum_{i=1}^{4n} H_i = F_{2n}K_{2n+2},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} K_i = L_{2n-1}K_{2n+1}, \qquad \sum_{i=1}^{4n} K_i = 5F_{2n}H_{2n+2}.$$

The arithmetical trick described above derives from the first formula of part (i) of the theorem with $n = 3$. For $n = 4$, it would say that the sum of the first 14 integers in the sequence is divisible by the ninth number in the sequence, and so on.

The proof of Theorem 1 depends on the following well-known results which we state for completeness.

Lemma 1: For $n \geqslant 1$,

$$H_n = aF_{n-2} + bF_{n-1} \quad \text{and} \quad K_n = aL_{n-2} + bL_{n-1}.$$

Lemma 2: For $n \geqslant 1$,

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \quad \text{and} \quad \sum_{i=1}^{n} L_i = L_{n+2} - 3.$$

Lemma 3: For integers r and s,

$$\text{(i)} \quad F_{r+2s} - F_r = \begin{cases} F_s L_{r+s} & s \text{ even,} \\ L_s F_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(ii)} \quad L_{r+2s} - L_r = \begin{cases} 5F_s F_{r+s} & s \text{ even,} \\ L_s L_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iii)} \quad F_{r+2s} + F_r = \begin{cases} L_s F_{r+s} & s \text{ even,} \\ F_s L_{r+s} & s \text{ odd,} \end{cases}$$

$$\text{(iv)} \quad L_{r+2s} + L_r = \begin{cases} L_s L_{r+s} & s \text{ even,} \\ 5F_s F_{r+s} & s \text{ odd.} \end{cases}$$

Note that Lemmas 1 and 2 are easily proved by induction and that Lemma 3 follows from Binet's formulas. Alternatively, Lemmas 1 and 2 follow from (7) and (6), page 456 of [2] for suitable choices of p and q, and Lemma 3 follows from (5)-(12), page 115 of [1] by setting $r = n - k$ and $s = k$. In fact, Theorem 1 can also be deduced from (6), page 456 of [2] and Lemma 3. However, for ease of reading, we give an independent proof.

Proof of Theorem 1: Since all the arguments are similar, we prove only part (iv). By Lemmas 1, 2, and 3,

$$\begin{aligned}
\sum_{i=1}^{4n} K &= \sum_{i=1}^{4n} (aL_{i-2} + bL_{i-1}) \\
&= aL_{-1} + aL_0 + a\sum_{i=3}^{4n} L_{i-2} + bL_0 + b\sum_{i=2}^{4n} L_{i-1} \\
&= -a + 2a + a(L_{4n} - 3) + 2b + b(L_{4n+1} - 3) \\
&= a(L_{4n} - L_0) + b(L_{4n+1} - L_1) \\
&= 5aF_{2n}^2 + 5bF_{2n}F_{2n+1} \\
&= 5F_{2n}(aF_{2n} + bF_{2n+1}) \\
&= 5F_{2n}H_{2n+2}
\end{aligned}$$

as claimed.

Setting $a = b = 1$, we obtain the following immediate corollary to Theorem 1.

Corollary 1: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} F_i = L_{2n-1}F_{2n+1}, \qquad \sum_{i=1}^{4n} F_i = F_{2n}L_{2n+2},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} L_i = L_{2n-1}L_{2n+1}, \qquad \sum_{i=1}^{4n} L_i = 5F_{2n}F_{2n+2}.$$

Now Lemma 1 and Theorem 1 suggest a further generalization. Define the sequences P, Q, R, S, T, U, V, and W for $n \geqslant 1$ by

$$\begin{aligned} P_n &= aF_{n-2} + bL_{n-1}, & Q_n &= aL_{n-2} + bF_{n-1}, \\ R_n &= aL_{n-2} + 5bF_{n-1}, & S_n &= 5aL_{n-2} + bF_{n-1}, \\ T_n &= aF_{n-2} + 5bL_{n-1}, & U_n &= 5aF_{n-2} + bL_{n-1}, \\ V_n &= aL_{n-2} + 5^2bF_{n-1}, & W_n &= 5^2aF_{n-2} + bL_{n-1}. \end{aligned}$$

Then the following results hold.

Theorem 2: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} P_i = L_{2n-1}P_{2n+1}, \qquad \sum_{i=1}^{4n} P_i = F_{2n}R_{2n+2},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} Q_i = L_{2n-1}Q_{2n+1}, \qquad \sum_{i=1}^{4n} Q_i = F_{2n}U_{2n+2},$$

$$\text{(iii)} \quad \sum_{i=1}^{4n-2} R_i = L_{2n-1}R_{2n+1}, \qquad \sum_{i=1}^{4n} R_i = 5F_{2n}P_{2n+2},$$

$$\text{(iv)} \quad \sum_{i=1}^{4n-2} S_i = L_{2n-1}S_{2n+1}, \qquad \sum_{i=1}^{4n} S_i = F_{2n}W_{2n+2},$$

$$\text{(v)} \quad \sum_{i=1}^{4n-2} T_i = L_{2n-1}T_{2n+1}, \qquad \sum_{i=1}^{4n} T_i = F_{2n}V_{2n+2},$$

$$\text{(vi)} \quad \sum_{i=1}^{4n-2} U_i = L_{2n-1}U_{2n+1}, \qquad \sum_{i=1}^{4n} U_i = 5F_{2n}Q_{2n+2},$$

$$\text{(vii)} \quad \sum_{i=1}^{4n-2} V_i = L_{2n-1}V_{2n+1}, \qquad \sum_{i=1}^{4n} V_i = 5F_{2n}T_{2n+2},$$

$$\text{(viii)} \quad \sum_{i=1}^{4n-2} W_i = L_{2n-1}W_{2n+1}, \qquad \sum_{i=1}^{4n} W_i = 5F_{2n}S_{2n+2}.$$

We omit the proof, since it is similar to that of Theorem 1.

3. MORE GENERAL RESULTS

We may generalize the results of Section 2 as follows. Define the sequences $\{f_n\}_{n\geqslant 0} = \{f_n(x)\}_{n\geqslant 0}$ and $\{\ell_n\}_{n\geqslant 0} = \{\ell_n(x)\}_{n\geqslant 0}$ by

$$f_0 = 0,\ f_1 = 1,\ f_{n+1} = af_n + f_{n-1}$$

and

$$\ell_0 = 2,\ \ell_1 = a,\ \ell_{n+1} = a\ell_n + \ell_{n-1},$$

where $a = a(x)$ is an arbitrary function of x. Then it is easily shown, as with the Fibonacci and Lucas sequences, that

$$f_n = \frac{\rho^n - \sigma^n}{\sqrt{a^2 + 4}} \tag{1}$$

and

$$\ell_n = \rho^n + \sigma^n \tag{2}$$

for all n where

$$\rho = \frac{a + \sqrt{a^2 + 4}}{2} \quad \text{and} \quad \sigma = \frac{a - \sqrt{a^2 + 4}}{2}.$$

Also,

$$\sum_{i=1}^{n} f_i = \frac{f_{n+1} + f_n - 1}{a}, \tag{3}$$

$$\sum_{i=1}^{n} \ell_i = \frac{\ell_{n+1} + \ell_n - a - 2}{a}, \tag{4}$$

$$f_{-1} = 1 \quad \text{and} \quad \ell_{-1} = -a. \tag{5}$$

In addition, we have the following generalization of Lemma 3.

Lemma 4: For integers r and s,

$$\text{(i)} \quad f_{r+2s} - f_r = \begin{cases} f_s \ell_{r+s} & s \text{ even}, \\ \ell_s f_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(ii)} \quad \ell_{r+2s} - \ell_r = \begin{cases} (a^2 + 4) f_s f_{r+s} & s \text{ even}, \\ \ell_s \ell_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(iii)} \quad f_{r+2s} + f_r = \begin{cases} \ell_s f_{r+s} & s \text{ even}, \\ f_s \ell_{r+s} & s \text{ odd}, \end{cases}$$

$$\text{(iv)} \quad \ell_{r+2s} + \ell_r = \begin{cases} \ell_s \ell_{r+s} & s \text{ even}, \\ (a^2 + 4) f_s f_{r+s} & s \text{ odd}. \end{cases}$$

Equations (1), (2), (3), and (4) can all be proved by induction, and Lemma 4 follows as before from the Binet formulas (1) and (2). Alternatively, (1) and (2) are essentially special cases of (53) and (54), page 119 of [1] and Lemma 4 is, in the same sense, a special case of (56)-(63) of [1].

If we now define the sequences h and k by

$$h_1 = c,\ h_2 = d,\ h_{n+1} = ah_n + h_{n-1} \tag{6}$$

and

$$k_1 = -ac + 2d,\ k_2 = ad + 2c,\ k_{n+1} = ak_n + k_{n-1}, \tag{7}$$

where $c = c(x)$ and $d = d(x)$ are also arbitrary functions of x, then it can be shown by induction that

$$h_n = cf_{n-2} + df_{n-1} \tag{8}$$

and

$$k_n = c\ell_{n-2} + d\ell_{n-1} \tag{9}$$

for all n. Finally, by analogy with Section 2, we define the sequences p, q, r, s, t, u, v, and w by

$$\begin{aligned}
p_n &= cf_{n-2} + d\ell_{n-1}\\
q_n &= c\ell_{n-2} + df_{n-1}\\
r_n &= c\ell_{n-2} + (a^2 + 4)df_{n-1}\\
s_n &= (a^2 + 4)c\ell_{n-2} + df_{n-1}\\
t_n &= cf_{n-2} + (a^2 + 4)d\ell_{n-1}\\
u_n &= (a^2 + 4)cf_{n-2} + d\ell_{n-1}\\
v_n &= c\ell_{n-2} + (a^2 + 4)^2 df_{n-1}\\
w_n &= (a^2 + 4)^2 cf_{n-2} + d\ell_{n-1}
\end{aligned}$$

for all n. Then, as before, we have the following result that generalizes both Theorem 1 and Theorem 2.

Theorem 3: For $n \geqslant 1$,

(i) $\displaystyle\sum_{i=1}^{4n-2} h_i = \frac{\ell_{2n-1}(h_{2n-1} + h_{2n})}{a}, \qquad \sum_{i=1}^{4n} h_i = \frac{f_{2n}(k_{2n+1} + k_{2n})}{a},$

(ii) $\displaystyle\sum_{i=1}^{4n-2} k_i = \frac{\ell_{2n-1}(k_{2n-1} + k_{2n})}{a}, \qquad \sum_{i=1}^{4n} k_i = \frac{(a^2 + 4)f_{2n}(h_{2n+1} + h_{2n})}{a},$

(iii) $\displaystyle\sum_{i=1}^{4n-2} p_i = \frac{\ell_{2n-1}(p_{2n-1} + p_{2n})}{a}, \qquad \sum_{i=1}^{4n} p_i = \frac{f_{2n}(r_{2n+1} + r_{2n})}{a},$

(iv) $\displaystyle\sum_{i=1}^{4n-2} q_i = \frac{\ell_{2n-1}(q_{2n-1} + q_{2n})}{a}, \qquad \sum_{i=1}^{4n} q_i = \frac{f_{2n}(u_{2n+1} + u_{2n})}{a},$

(v) $\displaystyle\sum_{i=1}^{4n-2} r_i = \frac{\ell_{2n-1}(r_{2n-1} + r_{2n})}{a}, \qquad \sum_{i=1}^{4n} r_i = \frac{(a^2 + 4)f_{2n}(p_{2n+1} + p_{2n})}{a},$

(vi) $\displaystyle\sum_{i=1}^{4n-2} s_i = \frac{\ell_{2n-1}(s_{2n-1} + s_{2n})}{a}, \qquad \sum_{i=1}^{4n} s_i = \frac{f_{2n}(w_{2n+1} + w_{2n})}{a},$

(vii) $\displaystyle\sum_{i=1}^{4n-2} t_i = \frac{\ell_{2n-1}(t_{2n-1} + t_{2n})}{a}, \qquad \sum_{i=1}^{4n} t_i = \frac{f_{2n}(v_{2n+1} + v_{2n})}{a},$

$$\text{(viii)} \quad \sum_{i=1}^{4n-2} u_i = \frac{\ell_{2n-1}(u_{2n-1} + u_{2n})}{a}, \qquad \sum_{i=1}^{4n} u_i = \frac{(a^2+4)f_{2n}(q_{2n+1} + q_{2n})}{a},$$

$$\text{(ix)} \quad \sum_{i=1}^{4n-2} v_i = \frac{\ell_{2n-1}(v_{2n-1} + v_{2n})}{a}, \qquad \sum_{i=1}^{4n} v_i = \frac{(a^2+4)f_{2n}(t_{2n+1} + t_{2n})}{a},$$

$$\text{(x)} \quad \sum_{i=1}^{4n-2} w_i = \frac{\ell_{2n-1}(w_{2n-1} + w_{2n})}{a}, \qquad \sum_{i=1}^{4n} w_i = \frac{(a^2+4)f_{2n}(s_{2n+1} + s_{2n})}{a}.$$

Proof: The proofs of these formulas are all similar to those of Theorem 1 and require the use of (3), (4), and Lemma 4 in the obvious places. To illustrate, we prove the first result in (i). Since $f_0 = 0$, we have that

$$\begin{aligned}
\sum_{i=1}^{4n-2} h_i &= \sum_{i=1}^{4n-2} (cf_{i-2} + df_{i-1}) = cf_{-1} + c\sum_{i=3}^{4n-2} f_{i-2} + d\sum_{i=2}^{4n-2} f_{i-1} \\
&= c + c\,\frac{f_{4n-3} + f_{4n-4} - 1}{a} + d\,\frac{f_{4n-2} + f_{4n-3} - 1}{a} \\
&= \frac{c(f_{4n-3} + f_{4n-4} + a - 1) + d(f_{4n-3} + f_{4n-3} - 1)}{a} \\
&= \frac{c(f_{4n-3} - f_1 + f_{4n-4} + f_2) + d(f_{4n-2} - f_0 + f_{4n-3} - f_1)}{a} \\
&= \frac{c(f_{2n-2}\ell_{2n-1} + f_{2n-3}\ell_{2n-1}) + d(f_{2n-1}\ell_{2n-1} + f_{2n-2}\ell_{2n-1})}{a} \\
&= \frac{\ell_{2n-1}[(cf_{2n-2} + df_{2n-1}) + (cf_{2n-3} + df_{2n-2})]}{a} \\
&= \frac{\ell_{2n-1}(h_{2n} + h_{2n-1})}{a}.
\end{aligned}$$

The formulas in Theorem 3 are still neat and tidy though not so simple as those in Theorems 1 and 2. The difficulty is that $H_{2n} + H_{2n-1} = H_{2n+1}$ in Theorem 1, whereas here we require $h_{2n} + ah_{2n-1} = h_{2n+1}$. Of course, if $a = 1$, the results coincide.

4. STILL MORE GENERAL RESULTS

It is natural to ask if the results can be generalized even further. Indeed, it would be reasonable to define sequences $\{\bar{f}_n\}_{n\geq 0} = \{\bar{f}_n(x)\}_{n\geq 0}$ and $\{\bar{\ell}_n\}_{n\geq 0} = \{\bar{\ell}_n(x)\}_{n\geq 0}$ by

$$\bar{f}_0 = 0,\ \bar{f}_1 = 1,\ \bar{f}_{n+1} = a\bar{f}_n + b\bar{f}_{n-1}$$

and

$$\bar{\ell}_0 = 2,\ \bar{\ell}_1 = a,\ \bar{\ell}_{n+1} = a\bar{\ell}_n + b\bar{\ell}_{n-1},$$

where $a = a(x)$ and $b = b(x)$ are arbitrary functions of x. Setting

$$\bar{\rho} = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \bar{\sigma} = \frac{a - \sqrt{a^2 + 4b}}{2},$$

we obtain as before (see [1], p. 119),

$$\bar{f}_n = \frac{\bar{\rho}^n - \bar{\sigma}^n}{\sqrt{a^2 + 4b}}, \tag{18}$$

$$\bar{\ell}_n = \bar{\rho}^n - \bar{\sigma}^n, \tag{19}$$

$$\sum_{i=1}^{n} \bar{f}_i = \frac{\bar{f}_{n+1} + b\bar{f}_n - 1}{a + b - 1}, \tag{20}$$

$$\sum_{i=1}^{n} \bar{\ell}_i = \frac{\bar{\ell}_{n+1} + b\bar{\ell}_n - a - 2b}{a + b - 1}, \tag{21}$$

and the following lemma.

Lemma 5: For integers r and s,

(i) $\bar{f}_{r+2s} - b^s\bar{f}_r = \begin{cases} \bar{f}_s\bar{\ell}_{r+s} & s \text{ even}, \\ \bar{\ell}_s\bar{f}_{r+s} & s \text{ odd}, \end{cases}$

(ii) $\bar{\ell}_{r+2s} - b^s\bar{\ell}_r = \begin{cases} (a^2 + 4b)\bar{f}_s\bar{f}_{r+s} & s \text{ even}, \\ \bar{\ell}_s\bar{\ell}_{r+s} & s \text{ odd}, \end{cases}$

(iii) $\bar{f}_{r+2s} + b^s\bar{f}_r = \begin{cases} \bar{\ell}_s\bar{f}_{r+s} & s \text{ even}, \\ \bar{f}_s\bar{\ell}_{r+s} & s \text{ odd}, \end{cases}$

(iv) $\bar{\ell}_{r+2s} + b^s\bar{\ell}_r = \begin{cases} \bar{\ell}_s\bar{\ell}_{r+s} & s \text{ even}, \\ (a^2 + 4b)\bar{f}_s\bar{f}_{r+s} & s \text{ odd}. \end{cases}$

Continuing, if we define $\bar{h}_i$ and $\bar{k}_i$ by

$$\bar{h}_1 = c,\ \bar{h}_2 = d,\ \bar{h}_{n+1} = a\bar{h}_n + b\bar{h}_{n-1} \tag{22}$$

and

$$\bar{k}_1 = 2d - ac,\ \bar{k}_2 = ad + 2bc,\ \bar{k}_{n+1} = a\bar{k}_n + b\bar{k}_{n-1} \tag{23}$$

where $c = c(x)$ and $d = d(x)$ as above, we prove as before that

$$\bar{h}_n = bc\bar{f}_{n-2} + d\bar{f}_{n-1} \tag{24}$$

and

$$\bar{k}_n = bc\bar{\ell}_{n-2} + d\bar{\ell}_{n-1}. \tag{25}$$

If, by analogy with (10)-(17), we now define sequences $\bar{p}_n$, $\bar{q}_n$, $\bar{r}_n$, $\bar{s}_n$, $\bar{t}_n$, $\bar{u}_n$, $\bar{v}_n$, and $\bar{w}_n$ by

$$\bar{p}_n = bc\bar{f}_{n-2} + d\bar{\ell}_{n-1}, \tag{26}$$

$$\bar{q}_n = bc\bar{\ell}_{n-2} + d\bar{f}_{n-1}, \tag{27}$$

$$\bar{r}_n = bc\bar{\ell}_{n-2} + (a^2 + 4b)d\bar{f}_{n-1}, \tag{28}$$

$$\bar{s}_n = (a^2 + 4b)bc\bar{\ell}_{n-2} + d\bar{f}_{n-1}, \tag{29}$$

$$\bar{t}_n = bc\bar{f}_{n-2} + (a^2 + 4b)d\bar{\ell}_{n-1}, \tag{30}$$

$$\bar{u}_n = (a^2 + 4b)bc\bar{f}_{n-2} + d\bar{\ell}_{n-1}, \tag{31}$$

$$\bar{v}_n = bc\bar{\ell}_{n-2} + (a^2 + 4b)\, d\bar{f}_{n-1}, \tag{32}$$

and

$$\bar{w}_n = (a^2 + 4b)^2 bc\bar{f}_{n-2} + d\bar{\ell}_{n-1}, \tag{33}$$

we can then prove the following theorems that contain all the preceding results as special cases. Of course, the formulas are less elegant, but they still exhibit a nice symmetry.

<u>Theorem 4</u>: For $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} \bar{f}_i + \frac{1 - b^{2n-1}}{a + b - 1} = \frac{\bar{\ell}_{2n-1}(\bar{f}_{2n} + b\bar{f}_{2n-1})}{a + b - 1},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n} \bar{f}_i + \frac{1 - b^{2n}}{a + b - 1} = \frac{\bar{f}_{2n}(\bar{\ell}_{2n+1} + b\bar{\ell}_{2n})}{a + b - 1},$$

$$\text{(iii)} \quad \sum_{i=1}^{4n-2} \bar{\ell}_i - \frac{(a + 2b)(1 - b^{2n-1})}{a + b - 1} = \frac{\bar{\ell}_{2n-1}(\bar{\ell}_{2n} + b\bar{\ell}_{2n-1})}{a + b - 1},$$

$$\text{(iv)} \quad \sum_{i=1}^{4n} \bar{\ell}_i - \frac{(a + 2b)(1 - b^{2n})}{a + b - 1} = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{f}_{2n+1} + b\bar{f}_{2n})}{a + b - 1}.$$

The proof is similar to that of Theorem 5 and will be omitted.

We note that Theorem 4 specializes to Corollary 1 if we set $a = b = c = d = 1$.

<u>Theorem 5</u>: Let

$$A = \frac{c + d - ac}{a + b - 1}, \qquad B = \frac{c(2b + a^2 - a) + d(2 - a)}{a + b - 1},$$

$$C = \frac{c(1 - a) + d(2 - a)}{a + b - 1}, \qquad D = \frac{c(2b + a^2 - a) + d}{a + b - 1},$$

$$E = \frac{c(2b + a^2 - a) + d(a^2 + 4b)}{a + b - 1}, \qquad F = \frac{c(a^2 + 4b)(2b + a^2 - a) + d}{a + b - 1},$$

$$G = \frac{c(1 - a) + d(a^2 + 4b)(2 - a)}{a + b - 1}, \qquad H = \frac{c(1 - a)(a^2 + 4b) + d(2 - a)}{a + b - 1},$$

$$I = \frac{c(2b + a^2 - a) + d(a^2 + 4b)}{a + b - 1}, \qquad J = \frac{c(1 - a)(a^2 + 4b)^2 + d(2 - a)}{a + b - 1}.$$

Then, for $n \geqslant 1$,

$$\text{(i)} \quad \sum_{i=1}^{4n-2} \bar{h}_i + A(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{h}_{2n} + b\bar{h}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{h}_i + A(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{k}_{2n+1} + b\bar{k}_{2n})}{+ \quad - 1},$$

$$\text{(ii)} \quad \sum_{i=1}^{4n-2} \bar{k}_i + B(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{k}_{2n} + b\bar{k}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{k}_i + B(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{h}_{2n+1} + b\bar{h}_{2n})}{a + b - 1},$$

$$\text{(iii)} \quad \sum_{i=1}^{4n-2} \bar{p}_i + C(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{p}_{2n} + b\bar{p}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{p}_i + C(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{r}_{2n+1} + b\bar{r}_{2n})}{a + b - 1},$$

$$\text{(iv)} \quad \sum_{i=1}^{4n-2} \bar{q}_i + D(1 - b^{2n-1}) + \frac{\bar{\ell}_{2n-1}(\bar{q}_{2n} + b\bar{q}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{q}_i + D(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{u}_{2n+1} + b\bar{u}_{2n})}{a + b - 1},$$

$$\text{(v)} \quad \sum_{i=1}^{4n-2} \bar{r}_i + E(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{r}_{2n} + b\bar{r}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{r}_i + E(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{p}_{2n+1} + b\bar{p}_{2n})}{a + b - 1},$$

$$\text{(vi)} \quad \sum_{i=1}^{4n-2} \bar{s}_i + F(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{s}_{2n} + b\bar{s}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{s}_i + F(1 - b^{2n}) = \frac{\bar{f}_{2}(\bar{w}_{2n+1} + b\bar{w}_{2n})}{a + b - 1},$$

$$\text{(vii)} \quad \sum_{i=1}^{4n-2} \bar{t}_i + G(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{t}_{2n} + b\bar{t}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{t}_i + G(1 - b^{2n}) = \frac{\bar{f}_{2n}(\bar{v}_{2n+1} + b\bar{v}_{2n})}{a + b - 1},$$

$$\text{(viii)} \quad \sum_{i=1}^{4n-2} \bar{u}_i + H(1 - b^{2n-1}) = \frac{\ell_{2n-1}(\bar{u}_{2n} + b\bar{u}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{u}_i + H(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{q}_{2n+1} + b\bar{q}_{2n})}{a + b - 1},$$

(ix) $$\sum_{i=1}^{4n-2} \bar{v}_i + I(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{v}_{2n} + b\bar{v}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{v}_i + I(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{t}_{2n+1} + b\bar{t}_{2n})}{a + b - 1}$$

(x) $$\sum_{i=1}^{4n-2} \bar{w}_i + J(1 - b^{2n-1}) = \frac{\bar{\ell}_{2n-1}(\bar{w}_{2n} + b\bar{w}_{2n-1})}{a + b - 1},$$

$$\sum_{i=1}^{4n} \bar{w}_i + J(1 - b^{2n}) = \frac{(a^2 + 4b)\bar{f}_{2n}(\bar{s}_{2n+1} + b\bar{s}_{2n})}{a + b - 1}.$$

Proof: Again, since the proofs are similar, we prove only the first part of (ii). Since $\bar{f}_{-1} = 1/b$, $\bar{f}_0 = 0$, and $\bar{\ell}_0 = 2$, we have from (20), (21), and Lemma 5, that

$$\begin{aligned}
\sum_{i=1}^{4n-2} \bar{p}_i &= \sum_{i=1}^{4n-2} (bc\bar{f}_{i-2} + d\bar{\ell}_{i-1}) = bc\left(\frac{1}{b}\right) + bc\sum_{i=1}^{4n-2} \bar{f}_{i-2} + 2d + d\sum_{i=1}^{4n-2} \bar{\ell}_{i-1} \\
&= c + \frac{bc(\bar{f}_{4n-3} + b\bar{f}_{4n-4} - 1)}{a + b - 1} + 2d + \frac{d(\bar{\ell}_{4n-2} + b\bar{\ell}_{4n-3} - a - 2b)}{a + b - 1} \\
&= \frac{ac - c + bc\bar{f}_{4n-3} + b^2c\bar{f}_{4n-4}}{a + b - 1} + \frac{ad - 2d + d\bar{\ell}_{4n-2} + db\bar{\ell}_{4n-3}}{a + b - 1} \\
&= \frac{bc(\bar{f}_{4n-3} - b^{2n-2}\bar{f}_1) + b^2c(\bar{f}_{4n-4} + b^{2n-3}\bar{f}_2)}{a + b - 1} \\
&\quad + \frac{d(\bar{\ell}_{4n-2} - b^{2n-1}\bar{\ell}_0) + db(\bar{\ell}_{4n-3} + p^{2n-2}\bar{\ell}_1)}{a + b - 1} \\
&\quad + \frac{c(a - 1) + d(a - 2) + b^{2n-1}c(1 - a) + b^{2n-1}d(2 - a)}{a + b - 1} \\
&= \frac{bc\bar{f}_{2n-2}\bar{\ell}_{2n-1} + b^2c\bar{f}_{2n-3}\,\bar{\ell}_{2n-1} + d\,\bar{\ell}_{2n-1}^2 + db\bar{\ell}_{2n-2}\bar{\ell}_{2n-1}}{a + b - 1} \\
&\quad + \frac{[c(a - 1) + d(a - 2)][1 - b^{2n-1}]}{a + b - 1} \\
&= \frac{\bar{\ell}_{2n-1}[bc\bar{f}_{2n-2} + d\bar{\ell}_{2n-1} + b(bc\bar{f}_{2n-3} + d\bar{\ell}_{2n-2})]}{a + b - 1} \\
&\quad + \frac{[c(a - 1) + d(a - 2)][1 - b^{2n-1}]}{a + b - 1} \\
&= \frac{\bar{\ell}_{2n-1}(\bar{p}_{2n} + b\bar{p}_{2n-1})}{a + b - 1} + \frac{[c(a - 1) + d(a - 2)][1 - p^{2n-1}]}{a + b - 1}
\end{aligned}$$

by definition of $\bar{p}_n$. But this implies the desired result.

Of course, if $b = 1$, these yield the formulas of Theorem 3 as they should.

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