A NOTE ON THE SUMS OF FIBONACCI AND LUCAS POLYNOMIALS

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Recently, G. E. Bergum and V. E. Hoggatt, Jr. [1] have shown that

$$\sum_{n=0}^{\infty} F^{-1}_{2^n k}(x) = \frac{1}{F_k(x)} + \begin{cases} (\alpha^2(x) + 1)/\alpha(x)(\alpha^{2k}(x) - 1), & x > 0, \\ (\beta^2(x) + 1)/\beta(x)(\beta^{2k}(x) - 1), & x < 0, \end{cases} \tag{1}$$

where $\{F_k(x)\}_{k=1}^{\infty}$ is the sequence of Fibonacci polynomials, defined recursively by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{k+2}(x) = xF_{k+1}(x) + F_k(x),\ k \geqslant 1,$$

and $\alpha(x) = (x + \sqrt{x^2 + 4})/2$, $\beta(x) = (x - \sqrt{x^2 + 4})/2$. Evidently, for $x = 1$ it is the known formula for the Fibonacci numbers [2].

In this paper we give, by an elementary method, an extension of the result (1). Namely, we show that

$$\sum_{n=0}^{\infty} (-1)^{r^n k} \frac{F_{(r-1)r^n k}(x)}{F_{r^n k}(x) F_{r^{n+1} k}(x)} = \begin{cases} \beta^k(x)/F_k(x), & x > 0, \\ \alpha^k(x)/F_k(x), & x < 0. \end{cases} \tag{2}$$

Obviously, for $r = 2$, we obtain (1) from (2).

Furthermore, we find that

$$\sum_{n=0}^{\infty} \frac{2^n \beta^{2^n k}(x)}{L_{2^n k}(x)} = \begin{cases} \dfrac{\alpha(x)}{\alpha^2(x) + 1} \dfrac{\beta^k(x)}{F_k(x)}, & x > 0, \\ \dfrac{\alpha(x)}{\alpha^2(x) + 1} \dfrac{\alpha^k(x)}{F_k(x)}, & x < 0, \end{cases} \tag{3}$$

where $L_k(k)$ is the Lucas polynomial defined by $L_k(x) = F_{k+1}(x) + F_{k-1}(x)$.

From the identity

$$\sum_{r=0}^{n} \frac{x^{p^r} - x^{p^{r+1}}}{(1 - x^{p^r})(1 - x^{p^{r+1}})} = \frac{x - x^{p^{n+1}}}{(1 - x)(1 - x^{p^{n+1}})}$$

if we put $x = \beta^k(x)/\alpha^k(x)$ we obtain

$$\sum_{r=0}^{m} (-1)^{n^r k} \frac{F_{(n-1)n^r k}(x)}{F_{n^r k}(x) F_{n^{r+1} k}(x)} = (-1)^k \frac{F_{(n^{m+1} - 1)k}(x)}{F_k(x) F_{n^{m+1} k}(x)}. \tag{4}$$

Using the facts that $|\beta(x)/\alpha(x)| < 1$ if $x > 0$ and that $\beta(x)/\alpha(x) < -1$ if $x < 0$, from (4), when $m \to \infty$, we have (2).

Similarly, from

$$\sum_{r=0}^{\infty} \frac{2^r x^{2^r}}{1 + x^{2^r}} = \frac{x}{1 - x},$$

if we put $x = \beta^k(x)/\alpha^k(x)$, we find (3).

REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Infinite Series with Fibonacci and Lucas Polynomials." *The Fibonacci Quarterly 17*, no. 2 (1979):147-151.

2. E. Lucas. *Theorie des nombres*. Paris, 1890.

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