LONGEST SUCCESS RUNS AND FIBONACCI-TYPE POLYNOMIALS

ANDREAS N. PHILIPPOU & FROSSO S. MAKRI
University of Patras, Patras, Greece

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1. INTRODUCTION AND SUMMARY

Let L_n be the length of the longest run of successes in n ($\geqslant 1$) independent trials with constant success probability p $(0 < p < 1)$, and set $q = 1 - p$. In [3] McCarty assumed that $p = 1/2$ and found a formula for the tail probabilities $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$ in terms of the Fibonacci sequence of order k [see Remark 2.1 and Corollary 2.1(c)]. In this paper, we establish a complete generalization of McCarty's result by deriving a formula for $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$ for any $p \in (0, 1)$. Formulas are also given for $P(L_n \leqslant k)$ and $P(L_n = k)$ $(0 \leqslant k \leqslant n)$. Our formulas are given in terms of the multinomial coefficients and in terms of the Fibonacci-type polynomials of order k (see Lemma 2.1, Definition 2.1, and Theorem 2.1). As a corollary to Theorem 2.1, we find two enumeration theorems of Bollinger [2] involving, in his terminology, the number of binary numbers of length n that do not have (or do have) a string of k consecutive ones. We present these results in Section 2. In Section 3, we reconsider the waiting random variable N_k $(k \geqslant 1)$, which denotes the number of Bernoulli trials until the occurrence of the k^{th} consecutive success, and we state and prove a recursive formula for $P(N_k = n)$ $(n \geqslant k)$ which is very simple and useful for computational purposes (see Theorem 3.1). We also note an interesting relationship between L_n and N_k. Finally, in Section 4, we show that $\sum_{k=0}^{n} P(L_n = k) = 1$ and derive the probability generating function and factorial moments of L_n. A table of means and variances of L_n when $p = 1/2$ is given for $1 \leqslant n \leqslant 50$.

We end this section by mentioning that the proofs of the present paper depend on the methodology of [4] and some results of [4] and [6]. Unless otherwise explicitly specified, in this paper k and n are positive integers and x and t are positive reals.

2. LONGEST SUCCESS RUNS AND FIBONACCI-TYPE POLYNOMIALS

We shall first derive a formula for $P(L_n \leqslant k)$ by means of the methodology of Theorem 3.1 of Philippou and Muwafi [4].

Lemma 2.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials. Then

$$P(L_n \leqslant k) = p^n \sum_{i=0}^{k} \sum_{n_1, \ldots, n_{k+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}}, \quad 0 \leqslant k \leqslant n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_{k+1}$, such that $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$.

Proof: A typical element of the event $(L_n \leqslant k)$ is an arrangement

$$x_1 x_2 \ \ldots \ x_{n_1 + \cdots + n_{k+1}} \underbrace{ss \ \ldots \ s}_{i},$$

such that n_1 of the x's are $e_1 = f$, n_2 of the x's are $e_2 = sf$, ..., n_{k+1} of the x's are $e_{k+1} = \underbrace{ss\ldots s}_{k}f$, and $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$ $(0 \leqslant i \leqslant k)$. Fix $n_1, \ldots, n_{k+1}$ and i. Then the number of the above arrangements is

$$\binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}},$$

and each one of them has probability

$$\begin{aligned} &P(x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{ss\ldots s}_{i}) \\ &= [P\{e_1\}]^{n_1}[P\{e_2\}]^{n_2} \ldots [P\{e_{k+1}\}]^{n_{k+1}} P\{\underbrace{ss\ldots s}_{i}\} \\ &= p^n(q/p)^{n_1+\cdots+n_{k+1}}, \quad 0 \leqslant k \leqslant n, \end{aligned}$$

by the independence of the trials, the definition of e_j $(1 \leqslant j \leqslant k+1)$, and $P\{s\} = p$. Therefore,

$$\begin{aligned} &P(\text{all } x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{ss\ldots s}_{i};\ n_j\ (1 \leqslant j \leqslant k+1) \text{ and } i \text{ fixed}) \\ &= \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} p^n(q/p)^{n_1+\cdots+n_{k+1}}, \quad 0 \leqslant k \leqslant n. \end{aligned}$$

But n_j $(1 \leqslant j \leqslant k+1)$ are nonnegative integers which may vary, subject to the condition $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$. Furthermore, i may vary over the integers $0, 1, \ldots, k$. Consequently,

$$\begin{aligned} &P(L_n \leqslant k) \\ &= P\left(\text{all } x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{s\ldots s}_{i};\ n_j \geqslant 0 \ni \sum_{j=1}^{k+1} jn_j = n - i,\ 0 \leqslant i \leqslant n\right) \\ &= \sum_{i=0}^{k} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n-i}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} p^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_{k+1}}, \ 0 \leqslant k \leqslant n, \end{aligned}$$

which establishes the lemma.

The formula for $P(L_n \leqslant k)$ derived in Lemma 2.1 can be simplified by means of the Fibonacci-type polynomials of order k [6]. These polynomials, as well as the Fibonacci numbers of order k [4], have been defined for $k \geqslant 2$, and the need arises presently for a proper extension of them to cover the cases $k = 0$ and $k = 1$. We shall keep the terminology of [6] and [4] despite the extension.

Definition 2.1: The sequence of polynomials $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequence of Fibonacci-type polynomials of order k if $F_n^{(0)}(x) = 0$ $(n \geqslant 0)$, and for $k \geqslant 1$, $F_0^{(k)}(x) = 0$, $F_1^{(k)}(x) = 1$, and

$$F_n^{(k)}(x) = \begin{cases} x[F_{n-1}^{(k)}(x) + \cdots + F_1^{(k)}(x)] & \text{if } 2 \leqslant n \leqslant k+1 \\ x[F_{n-1}^{(k)}(x) + \cdots + F_{n-k}^{(k)}(x)] & \text{if } n \geqslant k+2. \end{cases}$$

Definition 2.2: The sequence $\{F_n^{(k)}\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $F_n^{(0)} = 0$ $(n \geqslant 0)$, and for $k \geqslant 1$, $F_0^{(k)} = 0$, $F_1^{(k)} = 1$, and

$$F_n^{(k)} = \begin{cases} F_{n-1}^{(k)} + \cdots + F_1^{(k)} & \text{if } 2 \leqslant n \leqslant k+1 \\ F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} & \text{if } n \geqslant k+2. \end{cases}$$

It follows from Definitions 2.1 and 2.2 that

$$F_n^{(k)}(1) = F_n^{(k)}, \ n \geqslant 0. \tag{2.1}$$

The following lemma is useful in proving Theorem 2.1 below.

Lemma 2.2: Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k $(k \geqslant 1)$. Then,

(a) $F_n^{(1)}(x) = x^{n-1}$, $n \geqslant 1$, and $F_n^{(k)}(x) = x(1+x)^{n-2}$, $2 \leqslant n \leqslant k+1$;

(b) $$F_{n+1}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} x^{n_1 + \cdots + n_k}, \ n \geqslant 0.$$

Proof: Part (a) of the lemma follows easily from Definition 2.1. For $k = 1$, the right-hand side of (b) becomes x^n, which equals $F_{n+1}^{(1)}(x)$ $(n \geqslant 0)$ by (a), so that (b) is true. For $k \geqslant 2$, (b) is true because of Theorem 2.1(a) of [6].

Remark 2.1: Definition 2.2, Lemma 2.2(a), and (2.1) imply that the Fibonacci sequence of order k $(k \geqslant 1)$ coincides with the k-bonacci sequence (as it is given in McCarty [3]).

We can now state and prove Theorem 2.1, which provides another formula for $P(L_n \leqslant k)$. The new formula is a simplified version of the one given in Lemma 2.1, and it is stated in terms of the multinomial coefficients as well as in terms of the Fibonacci-type polynomials of order k. Formulas are also given for $P(L_n = k)$ $(0 \leqslant k \leqslant n)$ and $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$.

Theorem 2.1: Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k, and denote by L_n the length of the longest run of successes in n $(\geqslant 1)$ Bernoulli trials. Then,

(a) $$P(L_n \leqslant k) = \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}}$$
$$= \frac{p^{n+1}}{q} F_{n+2}^{(k+1)}(q/p), \ 0 \leqslant k \leqslant n;$$

(b) $$P(L_n = k) = \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)], \ 0 \leqslant k \leqslant n;$$

(c) $$P(L_n \geqslant k) = 1 - \frac{p^{n+1}}{q} F_{n+2}^{(k)}(q/p), \ 1 \leqslant k \leqslant n.$$

Proof: (a) Lemma 2.1, Lemma 2.2(b) applied with $x = q/p$, and Definition 2.1 give

$$P(L_n \leqslant k) = p^n \sum_{i=0}^{k} F_{n+1-i}^{(k+1)}(q/p) = \frac{p^{n+1}}{q} F_{n+2}^{(k+1)}(q/p), \ 0 \leqslant k \leqslant n,$$
$$= \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}},$$

which was to be shown. In order to show (b), we first observe that

$$P(L_n = k) = P(L_n \leqslant k) - P(L_n \leqslant k - 1)$$

$$= \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)],\ 1 \leqslant k \leqslant n,$$

by (a). Next, we note that

$$P(L_n = 0) = P(L_n \leqslant 0) = \frac{p^{n+1}}{q} F_{n+2}^{(1)}(q/p) = \frac{p^{n+1}}{q}[F_{n+2}^{(1)}(q/p) - F_{n+2}^{(0)}(q/p)],$$

since $F_{n+2}^{(0)}(q/p) = 0$ by Definition 2.1. The last two relations show (b). Finally, (c) is also true, since

$$P(L_n \geqslant k) = 1 - P(L_n \leqslant k - 1) = 1 - \frac{p^{n+1}}{q} F_{n+2}^{(k)}(q/p),\ 1 \leqslant k \leqslant n, \text{ by (a).}$$

We now have the following obvious corollary to the theorem.

Corollary 2.1: Let $\{F_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k and let L_n be as in Theorem 2.1. Assume $p = 1/2$. Then

(a) $P(L_n \leqslant k) = F_{n+2}^{(k+1)}/2^n,\quad 0 \leqslant k \leqslant n;$

(b) $P(L_n = k) = [F_{n+2}^{(k+1)} - F_{n+2}^{(k)}]/2^n,\quad 0 \leqslant k \leqslant n;$

(c) $P(L_n \geqslant k) = 1 - F_{n+2}^{(k)}/2^n,\quad 1 \leqslant k \leqslant n.$

Remark 2.2: McCarty [3] showed Corollary 2.1(c) by different methods.

We now proceed to offer the following alternative formulation and proof of Theorems 3.1 and 3.2 of Bollinger [2].

Corollary 2.2: For any finite set A, denote by $N(A)$ the number of elements in A, and let p, L_n, and $\{F_n^{(k)}\}_{n=0}^{\infty}$ be as in Corollary 2.1. Then

(a) $N(L_n < k) = F_{n+2}^{(k)},\quad n \geqslant 1;$

(b) $N(L_n = k) = F_{n+2}^{(k+1)} - F_{n+2}^{(k)},\quad n \geqslant 1.$

Proof: (a) Corollary 2.1(a) and the classical definition of probability give

$$\frac{N(L_n < k)}{2^n} = P(L_n < k) = P(L_n \leqslant k - 1) = \frac{F_{n+2}^{(k)}}{2^n},\quad 1 \leqslant k \leqslant n + 1.$$

Furthermore, it is obvious that

$$N(L_n < 0) = 0 \quad \text{and} \quad N(L_n < k) = 2^n,\quad k \geqslant n + 2.$$

The last two relations and Lemma 2.2(a) establish (a). Part (b) follows from Corollary 2.1(b), by means of the classical definition of probability and Lemma 2.2(a), in an analogous manner.

3. WAITING TIMES AND LONGEST SUCCESS RUNS

Denote by N_k the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 2$). Shane [7], Turner [8], Philippou and Muwafi [4], and Uppuluri and Patil [9] have all obtained alternative formulas for $P(N_k = n)$ ($n \geqslant k$). Presently, we derive another one, which is very simple and quite useful for computational purposes.

Theorem 3.1: Let N_k be a random variable denoting the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 1$). Then

$$P(N_k = n) = \begin{cases} p^k, & n = k, \\ qp^k, & k + 1 \leqslant n \leqslant 2k, \\ P[N_k = n - 1] - qp^k P[N_k = n - 1 - k], & n \geqslant 2k + 1. \end{cases}$$

The proof will be based on the following lemma of [4] and [6]. (See also [5].)

Lemma 3.1: Let N_k be as in Theorem 3.1. Then

$$\text{(a)}\quad P(N_k = n) = p^n \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - k}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}\left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant k;$$

$$\text{(b)}\quad P(N_k \leqslant n) = 1 - \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n + 1}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}\left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant k.$$

Proof of Theorem 3.1: By simple comparison, (a) and (b) of Lemma 3.1 give

$$P(N_k \leqslant n) = 1 - \frac{1}{qp^k} P(N_k = n + 1 + k), \quad n \geqslant k,$$

which implies

$$\begin{aligned} P(N_k = n) &= qp^k[1 - P(N_k \leqslant n - k - 1)] = qp^k\left[1 - \sum_{i=k}^{n-k-1} P(N_k = i)\right] \\ &= P[N_k = n - 1] - qp^k P[N_k = n - 1 - k], \quad n \geqslant 2k + 1. \end{aligned} \tag{3.1}$$

Next,

$$\begin{aligned} P(N_k = n) &= p^n F_{n-k+1}^{(k)}(q/p), \quad n \geqslant k, \text{ by Lemma 3.1(a) and Lemma 2.2(b),} \\ &= p^n\left(\frac{q}{p}\right)\left(1 + \frac{q}{p}\right)^{n-k-1}, \quad k + 1 \leqslant n \leqslant 2k, \text{ by Lemma 2.2(a),} \\ &= qp^k, \quad k + 1 \leqslant n \leqslant 2k. \end{aligned} \tag{3.2}$$

Finally, we note that

$$P(N_k = k) = P\{\underbrace{ss \ldots s}_{k}\} = p^k. \tag{3.3}$$

Relations (3.1)-(3.3) establish the theorem.

Remark 3.1: An alternative proof of another version of Theorem 3.1, based on first principles, is given independently by Aki, Kuboki, and Hirano [1].

We end this section by noting the following relation between L_n and N_k.

Proposition 3.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials, and denote by N_k the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 1$). Then

$$P(L_n \geqslant k) = P(N_k \leqslant n).$$

Proof: It is an immediate corollary of Theorem 2.1 and Lemma 3.1(b).

4. GENERATING FUNCTION AND FACTORIAL MOMENTS OF L_n

In this section, we show that $\{P(L_n = k)\}_{k=0}^{n}$ is a probability distribution and derive the probability generating function and factorial moments of L_n. It should be noted that our present results are given in terms of finite sums of Fibonacci-type polynomials where the running index is the order of the polynomial. It is conceivable that they could be simplified, but we are not aware of any results concerning such sums, even for the Fibonacci sequence of order k. For the case $p = 1/2$, we give a table of the means and variances of L_n for $1 \leqslant n \leqslant 50$.

Proposition 4.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials, and denote its generating function by $g_n(t)$. Also, set

$$x^{(0)} = 1 \quad \text{and} \quad x^{(r)} = x(x-1)\ \dots\ (x-r+1), \quad r \geqslant 1.$$

Then

(a) $$\sum_{k=0}^{n} P(L_n = k) = 1;$$

(b) $$g_n(t) = t^n - (t-1)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p), \quad n \geqslant 1.$$

(c) $$E(L_n^{(r)}) = n^{(r)} - r\,\frac{p^{n+1}}{q}\sum_{k=r-1}^{n-1} k^{(r-1)} F_{n+2}^{(k+1)}(q/p), \quad 1 \leqslant r \leqslant n;$$

(d) $$E(L_n) = n - \frac{p^{n+1}}{q}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p), \quad n \geqslant 1;$$

(e) $$\sigma^2(L_n) = \frac{p^{n+1}}{q}\left[(2n-1)\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p) - 2\sum_{k=1}^{n-1} kF_{n+2}^{(k+1)}(q/p)\right] - \left[\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p)\right]^2, \quad n \geqslant 2.$$

Proof: (a) We observe that $F_{n+2}^{(0)}(q/p) = 0$, by Definition 2.1, and

$$F_{n+2}^{(n+1)}(q/p) = (q/p)[1 + (q/p)]^n = q/p^{n+1}, \text{ by Lemma 2.2(a).}$$

Then Theorem 2.1(b) gives

$$\sum_{k=0}^{n} P(L_n = k) = \sum_{k=0}^{n} \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)]$$

$$= \frac{p^{n+1}}{q}[F_{n+2}^{(n+1)}(q/p) - F_{n+2}^{(0)}(q/p)] = 1.$$

(b) By means of Theorem 2.1(b), Definition 2.1, and Lemma 2.2(a), we have

$$g_n(t) = E(t^{L_n}) = \sum_{k=0}^{n} t^k P(L_n = k)$$

$$= \sum_{k=0}^{n} t^k \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)]$$

$$= \frac{p^{n+1}}{q}\left[\sum_{k=0}^{n} t^k F_{n+2}^{(k+1)}(q/p) - \sum_{k=-1}^{n-1} t^{k+1} F_{n+2}^{(k+1)}(q/p)\right]$$

$$= \frac{p^{n+1}}{q}\left[t^n F_{n+2}^{(n+1)}(q/p) + \sum_{k=0}^{n-1}(t^k - t^{k+1}) F_{n+2}^{(k+1)}(q/p)\right]$$

$$= t^n \frac{p^{n+1}}{q} F_{n+2}^{(n+1)}(q/p) + (1 - t)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p)$$

$$= t^n - (t - 1)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p),\ n \geqslant 1.$$

(c) It can be seen from (b), by induction on r, that the r^{th} derivative of $g_n(t)$ is given by

$$\frac{\partial^r}{\partial t^r} g_n(t) = n^{(r)} t^{n-r} - r\frac{p^{n+1}}{q}\sum_{k=r-1}^{n-1} k^{(r-1)} t^{k-r+1} F_{n+2}^{(k+1)}(q/p)$$

$$- (t - 1)\frac{p^{n+1}}{q}\sum_{k=r}^{n-1} k^{(r)} t^{k-r} F_{n+2}^{(k+1)}(q/p),\ 1 \leqslant r \leqslant n.$$

The last relation and the formula

$$E(L_n^{(r)}) = \frac{\partial^r}{\partial t^r} g_n(t)\Big|_{t=1}$$

establish (c). Now (d) follows from (c) for $r = 1$. Finally, (e) follows from (c) by means of the relation

$$\sigma^2(L_n) = E(L_n^{(2)}) + E(L_n) - [E(L_n)]^2.$$

Corollary 4.1: Let L_n be as in Proposition 4.1 and assume $p = 1/2$. Then

(a) $g_n(t) = t^n - \dfrac{(t-1)}{2^n}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)},\ n \geqslant 1.$

(b) $E(L_n^{(r)}) = n^{(r)} - \dfrac{r}{2^n}\sum_{k=r-1}^{n-1} k^{(r-1)} F_{n+2}^{(k+1)},\ 1 \leqslant r \leqslant n;$

(c) $E(L_n) = n - \dfrac{1}{2^n}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)},\ n \geqslant 1.$

(d) $$\sigma^2(L_n) = \frac{2n-1}{2^n}\sum_{k=0}^{n-1}F_{n+2}^{(k+1)} - \frac{1}{2^{n-1}}\sum_{k=1}^{n-1}kF_{n+2}^{(k+1)} - \frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1}F_{n+2}^{(k+1)}\right]^2, \quad n \geqslant 2.$$

We conclude this paper by presenting a table of means and variances of L_n, when $p = 1/2$, for $1 \leqslant n \leqslant 50$.

n	$E(L_n)$	$\sigma^2(L_n)$	n	$E(L_n)$	$\sigma^2(L_n)$
1	.500000	.250000	26	4.090650	2.691060
2	1.000000	.500000	27	4.142980	2.713386
3	1.375000	.734375	28	4.193483	2.734376
4	1.687500	.964844	29	4.242285	2.754142
5	1.937500	1.183594	30	4.289496	2.772786
6	2.156250	1.381836	31	4.335215	2.790402
7	2.343750	1.553711	32	4.379535	2.807071
8	2.511719	1.702988	33	4.422539	2.822872
9	2.662109	1.829189	34	4.464300	2.837871
10	2.798828	1.938046	35	4.504889	2.852132
11	2.923828	2.031307	36	4.544370	2.865711
12	3.039063	2.112732	37	4.582799	2.878660
13	3.145752	2.184079	38	4.620233	2.891025
14	3.245117	2.247535	39	4.656719	2.902849
15	3.338043	2.304336	40	4.692306	2.914170
16	3.425308	2.355688	41	4.727035	2.925023
17	3.507553	2.402393	42	4.760948	2.935439
18	3.585327	2.445150	43	4.794080	2.945448
19	3.659092	2.484463	44	4.826468	2.955075
20	3.729246	2.520765	45	4.858143	2.964345
21	3.796131	2.554392	46	4.889137	2.973278
22	3.860043	2.585633	47	4.919477	2.981895
23	3.921239	2.614727	48	4.949192	2.990214
24	3.979944	2.641880	49	4.978305	2.998250
25	4.036356	2.667271	50	5.006842	3.006021

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SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

August 13-16, 1986
San Jose State University
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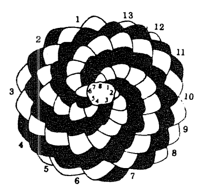

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CALL FOR PAPERS

The SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at San Jose State University, San Jose, CA, Aug. 13-16, 1986. This conference is sponsored jointly by The Fibonacci Association and San Jose State University.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are requested by February 15, 1986. Manuscripts are requested by April 1, 1986. Abstracts and manuscripts should be sent to the chairman of the local committee. Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by July 15, 1986. All talks should be limited to one hour.

For further information concerning the conference, please contact either of the following:

Professor G. E. Bergum, Editor
The Fibonacci Quarterly
Department of Mathematics
South Dakota State University
PO Box 2220
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Professor Calvin Long
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Washington State University
Pullman, WA 99163