MULTIPLE OCCURRENCES OF BINOMIAL COEFFICIENTS

CRAIG A. TOVEY
Georgia Institute of Technology, Atlanta, GA 30332

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I. INTRODUCTION

How many times can the same number appear in Pascal's triangle? After eliminating occurrences due to symmetry, $\binom{n}{k} = \binom{n}{n-k}$, and the uninteresting occurrences of $1 = \binom{n}{0}$ and $n = \binom{n}{1}$, the answer to this question is not clear. More precisely, if $1 < k \leqslant n/2$, we say that $\binom{n}{k}$ is a *proper* binomial coefficient. Are there integers that can be expressed in different ways as proper binomial coefficients?

Enumeration by hand or with a computer program produces some cases, given in Table 1. The smallest is 120, which equals

$$\binom{10}{3} \quad \text{and} \quad \binom{16}{2}.$$

Table 1. Small Multiple Occurrences of Binomial Coefficients

INTEGER	BINOMIAL COEFFICIENTS
120	$\binom{10}{3}$, $\binom{16}{2}$
210	$\binom{10}{4}$, $\binom{21}{2}$
1540	$\binom{22}{3}$, $\binom{56}{2}$
3003	$\binom{14}{6}$, $\binom{15}{5}$, $\binom{78}{2}$
7140	$\binom{36}{3}$, $\binom{120}{2}$
11628	$\binom{19}{5}$, $\binom{153}{2}$
24310	$\binom{17}{8}$, $\binom{221}{2}$

There is even an instance of a number, 3003, which can be expressed in three different ways. No clear pattern emerges; the cases just seem to be sprinkled among the binomial coefficients. We conjecture that, for any t, there exist infinitely many integers that may be expressed in t different (proper) ways as binomial coefficients.

Here we prove the conjecture for the case $t = 2$. The proof is constructive and depends in an unexpected way on the Fibonacci sequence.

II. THE CONSTRUCTION

We seek solutions to

$$\binom{n}{k} = \binom{n-1}{k+1}, \tag{1}$$

an especially tractable situation because it leads to a second-order equation. In particular, if (1) holds, then $n(k+1) = (n-k)(n-k-1)$. Let

$$x = n - k - 1. \tag{2}$$

Then $x(x+1) = n(n-x)$ so $n^2 - xn - (x^2 + x) = 0$ and

$$n = \frac{x + \sqrt{5x^2 + 4x}}{2} \tag{3}$$

(since n is positive). Integer solutions to (3) therefore lead to integer solutions to (1).

Since $5x^2 + 4x$ is even if and only if x is even, this means we must find integers x such that $5x^2 + 4x$ is a perfect square. Now x and $5x + 4$ have no common factors except possibly 2 or 4, so a natural slightly stronger condition would be that both x and $5x + 4$ be perfect squares. In other words, we need to find integers z such that $5z^2 + 4$ is a perfect square. These are given by the following lemma.

Lemma 1: Let F_j denote the Fibonacci sequence. Then, for all j,

$$(F_{j-1} + F_{j+1})^2 - 5F_j^2 = 4(-1)^j.$$

Proof: A straightforward calculation (see, e.g., [2], pp. 148-149) shows

$$(F_{j+1} + F_{j-1})^2 - 5F_j^2 = 4(F_{j-1}^2 + F_jF_{j-1} - F_j^2) = -4(F_j^2 + F_{j+1}F_j - F_{j+1}^2),$$

which yields the result by induction.

The lemma tells us that for any j even, $z = F_j$ gives the perfect square

$$5z^2 + 4 = (F_{j-1} + F_{j+1})^2.$$

This completes the construction.

Theorem 1: Let F_j denote the Fibonacci sequence. Then, for any even j, there exists a solution to (1), where $x = F_j^2$ and n and k are given by (2) and (3).

Remark: Letting L_j denote the Lucas sequence as usual, we can write this solution as

$$n = \frac{F_jL_j + F_j^2}{2}, \qquad k = \frac{F_jL_j - F_j^2}{2} - 1.$$

Theorem 2: Theorem 1 gives all solutions to (1).

Proof: It follows from the preceding discussion that any solution to (1) corresponds via (2) to some integer x such that $5x^2 + 4x$ is a perfect square. Let a (resp. b) be the number of times 2 divides $5x + 4$ (resp. x). If $a > 2$, then

$b = 2$ and, conversely, $b > 2$ implies $a = 2$. Since $5x + 4$ and x have no common factors except (possibly) 2 or 4, $(5x + 4)/2^a$ is a perfect square, as is $x/2^b$. Therefore, $a + b$ is even, so a and b are both even or both odd. In the former case, x and $5x + 4$ are perfect squares. We claim this leads precisely to the class of solutions given by Theorem 1. In the latter case, it follows that $a = b = 1$. Thus, we seek integers z such that $5z^2 + 2$ is a perfect square. We further claim that no such integers exist. The two claims can be shown to follows from the general theory of the so-called Pell equation (see, for example, [1] for the first claim, and [3, pp. 350-358] for the second claim). For completeness, we give a simple proof that does not rely on the general theory.

Let $\{A_n\}$ denote any sequence of positive numbers satisfying the recurrence $A_n + A_{n+1} = A_{n+2}$. The argument from Lemma 1 shows that, for all n,

$$(A_{n-1} + A_{n+1})^2 - 5A_n^2 = 4(A_{n-1}^2 + A_{n-1}A_n - A_n^2) = -(A_n + A_{n+2})^2 + 5A_{n+1}^2.$$

Therefore, given any solution z, y to $5z^2 + k = y^2$, we can construct smaller solutions by setting

$$A_i = z, \quad A_{i-1} = \frac{y - z}{2}, \quad A_{i+1} = \frac{y + z}{2},$$

and extending the sequence $\{A_n\}$ backward according to the recurrence

$$A_n + A_{n+1} = A_{n+2}.$$

[The solutions will be $z = A_j$, $y = A_{j-1} + A_{j+1}$, where $j \equiv i \pmod 2$, with

$$|k| = 4|A_n + A_nA_{n+1}^2 - A_{n+1}^2|, \text{ for all } n.]$$

Now, let (z, y) be any integer solution to $5z^2 + 4 = y^2$. Set

$$A_i = z \quad \text{and} \quad A_{i+1} = \frac{y + z}{2} \quad \text{(an integer).}$$

Then extend $\{A_n\}$ backward to get the solution corresponding to A_{i-2} and A_{i-1}. If $z \geqslant 3$, then

$$.61z \leqslant \frac{z\sqrt{5} - z}{2} \leqslant A_{i-1} = \frac{y - z}{2} \leqslant \frac{z\sqrt{5} + .5 - z}{2} \leqslant .72z,$$

whence $.28z \leqslant A_{i-2} \leqslant .39z$. Therefore, if $A_i \geqslant 3$, the solution corresponding to A_{i-2} is smaller. Repeatedly extend $\{A_n\}$ backward until $A_j < 3$. Since the only such integer solution is $(1, 3)$, z must have been a Fibonacci Number. This verifies the first claim. The second claim, that $5z^2 + 2 = y^2$ has no solutions, follows immediately from the fact that $y^2 \equiv 2 \pmod 5$ has none.

REFERENCES

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