NEWTON'S METHOD AND SIMPLE CONTINUED FRACTIONS

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1. INTRODUCTION

Let N be a positive integer that is not a perfect square. The Newton approximations to $\sqrt{N}$ will be obtained by fixing x_0 and setting

$$x_{n+1} = (x_n^2 + N)/(2x_n).$$

For example, one possible list of Newton approximations to $\sqrt{2}$ is

$$x_0 = 1,\ x_1 = 3/2,\ x_2 = 17/12,\ x_3 = 577/408,\ \ldots\ .$$

Let $(a_0, a_1, a_2, \ldots)$ represent the simple continued fraction for $\sqrt{N}$ with a_0, a_1, a_2, ... as partial quotients. Designate $c_n = p_n/q_n$, $(p_n, q_n) = 1$, as the n^{th} convergent of the continued fraction for $\sqrt{N}$. Thus, for example, $\sqrt{2} = (1, 2, 2, \ldots)$ has convergents

$$c_0 = 1,\ c_1 = 3/2,\ c_2 = 7/5,\ c_3 = 17/12,\ c_4 = 41/29,$$
$$c_5 = 99/70,\ c_6 = 239/169,\ c_7 = 577/408,\ \ldots\ .$$

Comparing the two lists of approximations, we see that each of the Newton approximations obtained in the manner above is a convergent of the continued fraction for $\sqrt{2}$; in fact, it appears that $x_n = c_{2^n-1}$. This is indeed the case and follows from Theorem 1 below (cf. [1, p. 468], [2], [3], [4]). We give a proof which appears to be simpler than those in the literature.

Theorem 1: If the continued fraction for $\sqrt{N}$ has period k, then for any positive integer m, Newton's method applied to c_{mk-1} results in c_{2mk-1}.

Proof: The s^{th} positive solution to the equation $x^2 - Ny^2 = \pm 1$ can be found in the following two ways:

(i) Write $(p_{k-1} + \sqrt{N}q_{k-1})^s$ in the form $u + \sqrt{N}v$, where u and v are integers; then $(x, y) = (u, v)$ is the s^{th} solution.

(ii) Calculate c_{sk-1}; then (p_{sk-1}, q_{sk-1}) is the s^{th} solution.

Letting $s = 2m$ gives

$$\begin{aligned} p_{2mk-1} + \sqrt{N}q_{2mk-1} &= [(p_{k-1} + \sqrt{N}q_{k-1})^m]^2 = (p_{mk-1} + \sqrt{N}q_{mk-1})^2 \\ &= p_{mk-1}^2 + Nq_{mk-1}^2 + \sqrt{N}(2p_{mk-1}q_{mk-1}) \end{aligned}$$

so that

$$p_{2mk-1}/q_{2mk-1} = (p_{mk-1}^2 + Nq_{mk-1}^2)/(2p_{mk-1}q_{mk-1}),$$

finishing the proof.

From Theorem 1, we see that whenever the continued fraction for $\sqrt{N}$ has period one, Newton's method applied to a convergent of the continued fraction for $\sqrt{N}$ results in a convergent. An identical result holds when the continued fraction for $\sqrt{N}$ has period two and follows as a corollary of the next theorem which we state without proof (cf. [3]).

Theorem 2: If the continued fraction for $\sqrt{N}$ has an even period $k = 2r$, then for any positive integer m, Newton's method applied to c_{mr-1} results in c_{2mr-1}.

We now know that if the continued fraction for $\sqrt{N}$ has period one or two, and if x_0 is a convergent of the continued fraction for $\sqrt{N}$, then we can conclude that all the successive approximations x_n are convergents of the continued fraction for $\sqrt{N}$. The following example shows that the conclusion is possible even when x_0 is rational but not a convergent. Let $N = 2$ and $x_0 = 2$; then we have $x_1 = 3/2$, $x_2 = 17/12$, $x_3 = 577/408$, ... which results in the same sequence we had with $x_0 = 1$. In the next two sections, we shall examine more closely the connection between Newton approximations and convergents in the cases when the continued fraction for $\sqrt{N}$ has period one or two.

2. CONTINUED FRACTION FOR $\sqrt{N}$ WITH PERIOD ONE

If the continued fraction for $\sqrt{N}$ has period one, we can tell for what rational x_0 the sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$ contains convergents and how many x_n are convergents. We note that if $x_0 = N/c_m$, then

$$x_1 = (x_0^2 + N)/(2x_0) = ([N/c_m]^2 + N)/(2N/c_m) = (c_m^2 + N)/(2c_m),$$

which is the same Newton approximation obtained if $x_0 = c_m$. Since x_{n+1} depends only on x_n (and N), we see that the entire sequence $\{x_n\}_{n=1}^{\infty}$ of Newton approximations to $\sqrt{N}$ is the same whether we begin with $x_0 = c_m$ or $x_0 = N/c_m$. This explains why we get the same Newton approximations to $\sqrt{2}$ when we begin with $x_0 = 1$ and when we begin with $x_0 = 2$.

Theorem 3: If the continued fraction for $\sqrt{N}$ has period one and if $\{x_n\}_{n=1}^{\infty}$ is the sequence of Newton approximations to $\sqrt{N}$ beginning with any rational number $x_0 \neq 0$, then either $\{x_n\}$ consists entirely of convergents or $\{x_n\}$ contains no convergents at all. Furthermore, $\{x_n\}$ consists entirely of convergents if and only if x_0 is a convergent or N times the reciprocal of a convergent.

Proof: We have already seen that if $x_0 = c_m$ or N/c_m for some nonnegative integer m, then $\{x_n\}_{n=1}^{\infty}$ consists entirely of convergents; therefore, it suffices to show that if x_0 is neither c_m nor N/c_m for any m, then $\{x_n\}$ contains no convergents. We begin with such an x_0 and prove by induction that every subsequent Newton approximation is of the same form. This is clearly the case if $x_0 < 0$, since for such an x_0 we have $\{x_n\}$ contains only negative numbers. Now consider $x_0 > 0$. Suppose that we have shown that x_n is neither c_m nor N/c_m for any m. Then

$$x_{n+1} = (x_n^2 + N)/(2x_n),$$

which is equivalent to

$$x_n^2 - 2x_{n+1}x_n + N = 0 \tag{1}$$

or

$$x_n = x_{n+1} \pm (x_{n+1}^2 - N)^{1/2}. \tag{2}$$

Assume $x_{n+1} = c_m$ for some nonnegative integer m. Since x_0 is rational and therefore real, so is x_n, whence, by (2), $x_{n+1} > \sqrt{N}$; this means that x_{n+1} must be an odd convergent. By Theorem 1, taking $k = 1$, we see that Newton's method applied to $c_{n/2}$ results in x_{n+1}. Consequently, Newton's method applied to $N/c_{n/2}$ also results in x_{n+1}. Since $\sqrt{N}$ is irrational, $c_{n/2} \neq N/c_{n/2}$. Hence, $c_{n/2}$ and $N/c_{n/2}$ must be the two distinct roots of (1) so that, contrary to the induction hypothesis, $x_n = c_{n/2}$ or $N/c_{n/2}$.

Assume now that $x_{n+1} = N/c_m$ for some m. Then (2) becomes

$$x_n = N/c_m \pm (\{N(Nq_m^2 - p_m^2)\}^{1/2}/p_m). \tag{3}$$

Since x_n is rational, we must have $\{N(Nq_m^2 - p_m^2)\}^{1/2}$ rational. But the continued fraction for $\sqrt{N}$ has period one, so that $Nq_m^2 - p_m^2 = \pm 1$, and therefore,

$$\{N(Nq_m^2 - p_m^2)\}^{1/2} = \{\pm N\}^{1/2},$$

which is not rational. Hence, $x_{n+1} \neq N/c_m$ for any m, completing the proof.

3. CONTINUED FRACTION FOR $\sqrt{N}$ WITH PERIOD TWO

When the continued fraction for $\sqrt{N}$ has period two, a theorem analogous to Theorem 3 does not exist. To see this, consider $N = 12$ and $x_0 = 6$. We have

$$\sqrt{12} = (3, 2, 6, 2, 6, \ldots),$$

with convergents 3, 7/2, 45/13, ..., so that x_0 is not a convergent. Also, $x_0 = 12/2$ so that x_0 is not $12/c_m$ for any m. But

$$(6^2 + 12)/(2 \cdot 6) = 12/3 = 12/c_0$$

which means, by an argument similar to that used at the beginning of Section 2, Newton's method applied twice to x_0 yields a convergent, namely $c_1 = 7/2$. We shall see, in fact, that there are infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and, for some rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence $\{x_n\}$ contains infinitely many convergents. On the other hand, we shall see that there are infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and, for any rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence $\{x_n\}$ contains no convergents. Before we begin, we note that some of the results of Section 2 carry over immediately into this section, namely Newton's method applied to c_m is identical to Newton's method applied to N/c_m, and the first part of the induction proof for Theorem 3 works here by using Theorem 2 rather than Theorem 1.

Theorem 4: Let S be the set of all $s = kx^2$ or $4kx^2$ where $x^2 - ky^2 = 1$ for some positive integers x, y, and k. If $N \in S$, then the continued fraction for $\sqrt{N}$ has period two and there is a rational x_0 not of the form c_m or N/c_m such that the sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$, beginning with x_0, contains infinitely many convergents. Also, if $N \notin S$ and the continued fraction for $\sqrt{N}$ has period two, then for any rational x_0 that is neither a c_m nor an N/c_m, the resulting sequence contains no convergents.

Proof: Let T be the set of all N such that the continued fraction for $\sqrt{N}$ has period two and, for any rational x_0 not of the type c_m or N/c_m, the resulting sequence $\{x_n\}$ of Newton approximations to $\sqrt{N}$, beginning with x_0, contains no convergent of the continued fraction for $\sqrt{N}$. Consider some N such that the continued fraction for $\sqrt{N}$ has period two. We show first that $N \notin T$ if and only $\{N(N - p_0^2)\}^{1/2}$ is rational. Assume $\{N(N - p_0^2)\}^{1/2}$ is rational. Set

$$x_0 = N/c_0 \pm (\{N(N - p_0^2)\}^{1/2}/p_0). \tag{4}$$

Since $q_0 = 1$, (4) is precisely (3) with $n = m = 0$. Thus, $x_1 = N/c_0$. Since the continued fraction for $\sqrt{N}$ has period two, there are positive integers a and b such that $b|2a$, $N = a^2 + (2a/b)$ and $\sqrt{N} = (a, b, 2a, b, 2a, \ldots)$, so the first two convergents of the continued fraction for $\sqrt{N}$ are a and $(ab + 1)/b$. Also, $x_1 = N/c_0 = (a^2 + 2a/b)/a = (ab + 2)/b$. Thus, x_1 is not a convergent. Therefore, x_0 is different from c_m and N/c_m for all m, but the sequence $\{x_n\}$ contains infinitely many convergents of the continued fraction for $\sqrt{N}$, namely all x_k for $k \geqslant 2$. Thus, $N \notin T$.

Now assume $\{N(N - p_0^2)\}^{1/2}$ is not rational. Suppose x_n is the n^{th} Newton approximation to $\sqrt{N}$ starting from some rational x_0 and is given by (2) and (3) where $x_{n+1} = N/c_m$ for some m. From (2) and the fact that x_n is rational, we have that $x_{n+1} > \sqrt{N}$ so that $c_m < \sqrt{N}$ and m is even. Thus,

$$p_m^2 - Nq_m^2 = p_0^2 - Nq_0^2 = p_0^2 - N,$$

so that by (3),

$$x_n = N/c_m \pm (\{N(N - p_0^2)\}^{1/2}/p_m),$$

which is not rational by assumption, giving a contradiction. The induction argument given in the proof of Theorem 3 now works here, and we may conclude that $N \in T$, which finishes what we first set out to show.

To complete the proof of the theorem we need only show that the continued fraction for $\sqrt{N}$ has period two and $\{N(N - p_0^2)\}^{1/2}$ is rational if and only if $N \in S$. Consider N such that the continued fraction for $\sqrt{N}$ has period two and write, as before, $N = a^2 + (2a/b)$ where $b|(2a)$. Assume that $\{N(N - p_0^2)\}^{1/2}$ is rational. We have $N - p_0^2 = N - a^2 = 2a/b$ so that $N(N - p_0^2) = N(2a/b) = d^2$ for some positive integer d. Then we consider two possible cases.

Case 1. b is odd.

Here $b|a$. Set $a_1 = a/b$ so that $N = b^2a_1^2 + 2a_1$. Therefore,

$$d^2 = 2a_1^2(b^2a_1 + 2). \tag{5}$$

Thus, a_1 is even and $(2a_1)|d$. Writing $a_1 = 2a_2$ and $d = 2a_1d_1$, (5) becomes

$$d_1^2 - a_2b^2 = 1,$$

and, therefore,

$$N = b^2a_1^2 + 2a_1 = 4a_2(b^2a_2 + 1) = 4a_2d_1^2.$$

Hence, $N \in S$.

Case 2. b is even.

Here $b = 2b_1$, where $b_1|a$, so that $a = b_1a_1$ and $d = a_1d_1$ for some integers a_1 and d_1 with

$$d_1^2 - a_1b_1^2 = 1. \tag{6}$$

We conclude that $N = b_1^2a_1^2 + a_1 = a_1d_1^2$ and, therefore, $N \in S$.

Now suppose $N \in S$. Then $N = 4kx^2$ or kx^2 for some positive integers x, y, and k such that $x^2 - ky^2 = 1$.

Case 1. $N = 4kx^2$.

Here set $a = 2ky$ and $b = y$. Then

$$a^2 + (2a/b) = 4k(ky^2 + 1) = 4kx^2 = N \quad \text{and} \quad b|(2a).$$

Also, $b \neq 2a$, since $y < 4ky$. Thus, the continued fraction for $\sqrt{N}$ has period two. Also, we get $N(N - p_0^2) = (4kx)^2$.

Case 2. $N = kx^2$.

Here set $a = ky$ and $b = 2y$. Then $a^2 + (2a/b) = N$ and $b|2a$, so that the continued fraction for $\sqrt{N}$ has period two (note that $b \neq 2a$ since $x^2 - y^2 \neq 1$). Also, $N(N - p_0^2) = (kx)^2$.

This completes the proof.

Corollary 1: If the continued fraction for $\sqrt{N}$ has period two and N is square-free, then $N \notin S$.

Proof: Suppose $N \in S$. Then $N = kx^2$ or $4kx^2$ for some positive integers x and k. Thus, $x = 1$ and $1 - ky^2 = 1$ for some positive integer y, giving a contradiction.

Corollary 2: If $N = (2d)^2 + 2$, where d is the denominator of an odd convergent of the continued fraction for $\sqrt{2}$, then $N \in S$. On the other hand, if $N = a^2 + 2$ for any positive integer a not twice the denominator of an odd convergent of the continued fraction for $\sqrt{2}$, then $N \notin S$. In particular, if N is odd and of the form $a^2 + 2$, then $N \notin S$.

Proof: Consider $N = a^2 + (2a/b)$, where $b = a$. From the proof of Theorem 4, we know that $N \in S$ if and only if $2N = N(2a/b) = d_1^2$ for some positive integer d_1 if and only if $a^2 + 2 = N = 2d_2^2$ for some positive integer d_2 if and only if $a = 2d$ for some positive integer d and $d_2^2 - 2d^2 = 1$ if and only if $N = a^2 + 2$, where $a = 2d$ and d_2/d is an odd convergent of the continued fraction for $\sqrt{2}$, which proves the first part of the corollary. The last statement follows from the observation that if N is odd, then a is odd.

Corollary 3: There exist infinitely many $N \in S$ and infinitely many N such that the continued fraction for $\sqrt{N}$ has period two and $N \notin S$.

Proof: Take N of the form $a^2 + 2$, and use Corollary 2.

Finally, we note that the only $N \in S$ less than 1000 are

12, 18, 48, 72, 147, 150, 240, 288, 405, 448, 578, 588, 600, 960.

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LETTER FROM THE EDITOR

The editor wishes to express his gratitude to those who have agreed to referee papers for *The Fibonacci Quarterly* during 1986. A complete list of these referees will be given in the May 1986 issue.

G. E. Bergum