SUMMATION OF RECIPROCAL SERIES OF NUMERICAL FUNCTIONS OF SECOND ORDER

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This paper is an extension of the results of G. E. Bergum and V. E. Hoggatt, Jr. [1] concerning the problem of summation of reciprocals of products of Fibonacci and Lucas polynomials. The method used here will also allow us to generalize some formulas of R. Backstrom [2] related to sums of reciprocal series of Fibonacci and Lucas numbers.

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The general numerical functions of second order which, following the notation of Horadam [3], we write as $\{w_n(a, b; p, q)\}$ may be defined by

$$w_n = pw_{n-1} - qw_{n-2},\ n \geqslant 2,\ w_0 = a,\ w_1 = b,$$

with

$$w_n = w_n(a, b; p, q),$$

where a and b are arbitrary integers.

We are interested in the sequences

$$u_n = w_n(0, 1; p, q) \tag{1}$$

and

$$v_n = w_n(2, p; p, q) \tag{2}$$

that can be expressed in the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geqslant 1, \tag{3}$$

and

$$v_n = \alpha^n + \beta^n, \quad n \geqslant 1, \tag{4}$$

where

$$\alpha = (p + \sqrt{p^2 - 4q})/2,\ \beta = (p - \sqrt{p^2 - 4q})/2,\ \alpha + \beta = p,\ \alpha\beta = q,$$

and $\alpha - \beta = \delta = \sqrt{\Delta}$.

Using (3) and (4), we obtain

$$2\alpha^n = v_n + \delta u_n$$

and

$$4\alpha^{m+n} = v_m v_n + \Delta u_m u_n + \delta(u_m v_n + u_m v_n),$$

from which it follows that

$$u_{s+r}v_s - u_s v_{s+r} = 2q^s u_r \tag{5}$$

and

$$v_{s+r}v_s - \Delta u_s u_{s+r} = 2q^s v_r. \tag{6}$$

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From relation (5), we have

$$\frac{v_s}{u_s} - \frac{v_{s+r}}{u_{s+r}} = 2q^s \frac{u_r}{u_s u_{s+r}}.$$

If we replace s here by s, $s + r$, $s + 2r$, ..., $s + (n - 1)r$, successively, and add the results, we obtain, due to the telescoping effect,

$$S_n(p, q; r, s) = \sum_{k=1}^{n} \frac{q^{(k-1)r}}{u_{s+(k-1)r}u_{s+kr}} = \left(\frac{v_s}{u_s} - \frac{v_{s+nr}}{u_{s+nr}}\right)\frac{1}{2q^s u_r} = \frac{u_{nr}}{u_r u_s u_{s+nr}}. \tag{7}$$

Similarly, again using (5), we also have

$$\sigma_n(p, q; r, s) = \sum_{k=1}^{n} \frac{q^{(k-1)r}}{v_{s+(k-1)r}v_{s+kr}} = \left(\frac{u_{s+nr}}{v_{s+nr}} - \frac{u_s}{v_s}\right)\frac{1}{2q^s u_r} = \frac{u_{nr}}{u_r v_s v_{s+nr}}. \tag{8}$$

Because

$$\lim_{n\to\infty} \frac{u_n}{u_{n+r}} = \begin{cases} \alpha^{-r}, & |\beta/\alpha| < 1 \\ \beta^{-r}, & |\alpha/\beta| < 1, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{u_n}{v_{n+r}} = \begin{cases} \alpha^{1-r}/(\alpha^2 - q), & |\beta/\alpha| < 1 \\ \beta^{1-r}/(\beta^2 - q), & |\alpha/\beta| < 1, \end{cases}$$

we obtain

$$S(p, q; r, s) = \sum_{k=1}^{\infty} \frac{q^{(k-1)r}}{u_{s+(k-1)r}u_{s+kr}} = \begin{cases} \dfrac{\alpha^{-s}}{u_r u_s}, & |\beta/\alpha| < 1 \\[2ex] \dfrac{\beta^{-s}}{u_r u_s}, & |\alpha/\beta| < 1, \end{cases} \tag{9}$$

$$\sigma(p, q; r, s) = \sum_{k=1}^{\infty} \frac{q^{(k-1)r}}{v_{s+(k-1)r}v_{s+kr}} = \begin{cases} \dfrac{\alpha^{1-s}}{\alpha^2 - q}\dfrac{1}{u_r v_s}, & |\beta/\alpha| < 1 \\[2ex] \dfrac{\beta^{1-s}}{\beta^2 - q}\dfrac{1}{u_r v_s}, & |\alpha/\beta| < 1. \end{cases} \tag{10}$$

In particular, with $r = s$, we have

$$S(p, q; r, r) = \begin{cases} \alpha^{r-2}\left(\dfrac{\alpha^2 - q}{\alpha^{2r} - q^r}\right)^2, & |\beta/\alpha| < 1 \\[2ex] \beta^{r-2}\left(\dfrac{\beta^2 - q}{\beta^{2r} - q^r}\right)^2, & |\alpha/\beta| < 1, \end{cases} \tag{11}$$

and

$$\sigma(p, q; r, r) = \begin{cases} \alpha^r/(\alpha^{4r} - q^{2r}), & |\beta/\alpha| < 1 \\ \beta^r/(\beta^{4r} - q^{2r}), & |\alpha/\beta| < 1. \end{cases} \tag{12}$$

3. SPECIAL CASES

It is not difficult to obtain the formulas of Bergum and Hoggatt from (9) and (10). Indeed, if we let $p = x$ and $q = -1$ in (1) and (2), these relations define the sequences of the Fibonacci polynomials $\{F_k(x)\}_{k=1}^{\infty}$ and the Lucas polynomials $\{L_k(x)\}_{k=1}^{\infty}$. In this case,

$$\alpha(x) = (x + \sqrt{x^2 + 4})/2, \quad \beta(x) = (x - \sqrt{x^2 + 4})/2,$$

where

$$-1 < \alpha(x) < 1 \quad \text{and} \quad \beta(x) > 1 \quad \text{when } x > 0,$$

$$0 < \alpha(x) < 1 \quad \text{and} \quad \beta(x) < 1 \quad \text{when } x < 0.$$

Hence, (9) and (10) become

$$S(x, -1; r, s) = \lim_{n \to \infty} S_n(x, -1; r, s) = \begin{cases} \dfrac{1}{\alpha^s(x)} \dfrac{1}{F_r(x)F_s(x)}, & x > 0, \\ \dfrac{1}{\beta^s(x)} \dfrac{1}{F_r(x)F_s(x)}, & x < 0, \end{cases} \tag{13}$$

and

$$\sigma(x, -1; r, s) = \lim_{n \to \infty} \sigma_n(x, -1; r, s) = \begin{cases} \dfrac{\alpha^{1-s}(x)}{1 + \alpha^2(x)} \dfrac{1}{F_r(x)L_s(x)}, & x > 0 \\ \dfrac{\beta^{1-s}(x)}{1 + \beta^2(x)} \dfrac{1}{F_r(x)L_s(x)}, & x < 0. \end{cases} \tag{14}$$

Comparing the results of Bergum and Hoggatt [1, p. 149, formulas (9) and (17)] with our (13) and (14) above, we find that

$$U(q, a, b, x) = (-1)^b F_k(x)F_q(x)S(x, -1; q, b) \tag{15}$$

and

$$V(q, a, b, x) = (-1)^b F_k(x)F_q(x)(x^2 + 4)\sigma(x, -1; q, b), \tag{16}$$

when $q = b - a + k$.

As particular cases, we give:

$$S(x, -1; 2, 2) = \sum_{k=1}^{\infty} \frac{1}{F_{2k}(x)F_{2(k+1)}(x)} = \begin{cases} \beta^2(x)/x^2, & x > 0, \\ \alpha^2(x)/x^2, & x < 0, \end{cases}$$

and

$$\sigma(x, -1; 2, 2) = \sum_{k=1}^{\infty} \frac{1}{L_{2k}(x)L_{2(k+1)}(x)} = \begin{cases} \alpha^2(x)/(\alpha^8(x) - 1), & x > 0, \\ \beta^2(x)/(\beta^8(x) - 1), & x < 0. \end{cases}$$

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Using the relations (5) and (6) with $u_{-n} = -q^{-n}u_n$ and $v_{-n} = q^{-n}v_n$, we have

$$v_{2r} - q^{r-s}v_{2s} = \Delta u_{r-s}u_{r+s}.$$

Then, by the method used to obtain (7), we have

$$\Delta \sum_{k=0}^{n} \frac{q^{kr}}{v_{(2k+1)r+2s} - q^{s+kr}v_r} = \frac{1}{u_s u_r}\frac{u_{(n+1)r}}{u_{s+(n+1)r}} \tag{17}$$

so that

$$\Delta \sum_{k=0}^{\infty} \frac{q^{s+kr}}{v_{(2k+1)r+2s} - q^{s+kr}v_r} = \begin{cases} \dfrac{\beta^s}{u_r u_s}, & |\beta/\alpha| < 1, \\[2ex] \dfrac{\alpha^s}{u_r u_s}, & |\alpha/\beta| < 1. \end{cases} \tag{18}$$

Similarly, from

$$v_{2r} + q^{r-s}v_{2s} = v_{r-s}v_{r+s},$$

using (8) we obtain

$$\sum_{k=0}^{n} \frac{q^{kr}}{v_{(2k+1)r+2s} + q^{s+kr}v_r} = \frac{1}{u_r v}\frac{u_{(n+1)r}}{v_{s+(n+1)r}} \tag{19}$$

or

$$\sum_{k=0}^{\infty} \frac{q^{s+kr}}{v_{(2k+1)r+2s} + q^{s+kr}v_r} = \begin{cases} \beta^{s+1}/(q - \beta^r)u_r v_s, & |\beta/\alpha| < 1, \\ \alpha^{s+1}/(q - \alpha^r)u_r v_s, & |\alpha/\beta| < 1. \end{cases} \tag{20}$$

In particular, if we put $p = -q = 1$ in (17)-(20), we obtain the formulas of Backstrom [2] concerning the Lucas numbers. These are

$$\sum_{k=0}^{n} \frac{1}{L_{(2k+1)r+2s} + L_r} = \begin{cases} \dfrac{1}{5F_r F_s}\dfrac{F_{(n+1)r}}{F_{(n+1)r+s}}, & s \text{ odd}, \\[2ex] \dfrac{1}{F_r L_s}\dfrac{F_{(n+1)r}}{L_{(n+1)r+s}}, & s \text{ even}, \end{cases}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{L_{(2k+1)r+2s} + L_r} = \begin{cases} \left(\dfrac{-1+\sqrt{5}}{2}\right)^s \dfrac{1}{5F_r F_s}, & s \text{ odd}, \\[2ex] \left(\dfrac{\sqrt{5}-1}{2}\right)^s \dfrac{1}{5F_r L_s}, & s \text{ even}, \end{cases}$$

where r is an even integer satisfying $-r \leqslant 2s \leqslant r - 2$.

We notice that, from

$$u_r^2 - q^{r-s}u_s^2 = u_{r-s}u_{r+s},$$

it follows that

$$\sum_{k=1}^{n} \frac{q^{2(n-1)r}}{u^2_{(2k-1)r+s} - q^{s+2kr}u_r^2} = S_n(p,\ q;\ 2r,\ s) \tag{21}$$

and

$$\sum_{k=1}^{\infty} \frac{q^{2(n-1)r}}{u^2_{(2k-1)r+s} - q^{s+2kr}u_r^2} = \begin{cases} \beta^s/u_{2r}u_s, & |\beta/\alpha| < 1, \\ \alpha^s/u_{2r}u_s, & |\alpha/\beta| < 1. \end{cases}$$

Similarly,

$$\Delta \sum_{k=1}^{n} \frac{q^{2(k-1)r}}{v^2_{(2k-1)r+s} - q^{s+2kr}v_r^2} = S_n(p,\ q;\ 2r,\ s).$$

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