ON A CLASS OF KNOTS WITH FIBONACCI INVARIANT NUMBERS

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This paper describes how a subclass of the rational knots* may be constructed sequentially, the knots in the sequence having 1, 2, ..., i, ... crossings. For these knots, the values of a certain knot invariant are Fibonacci numbers, the i^{th} knot in the sequence having invariant number F_i.

The knot invariant has a wide number of interpretations and properties, and some of these will be outlined, particularly in relation to knots in the constructed class.

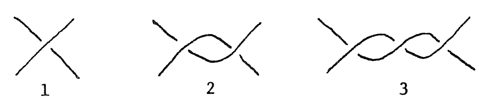
The class will be called the *Fibonacci knot-class*. A generalization of this class will be introduced and briefly discussed.

1. THE RATIONAL KNOTS

J. H. Conway [2] defines the notion of "integer tangle," and gives rules for combining integer tangles to form a large class of alternating knots which he calls *rational knots*. He develops operations by which all knots on a given number of crossings may be constructed and tested for equivalences.

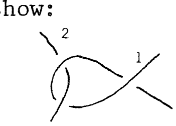
Conway's Notation and Construction of the Rational Knots

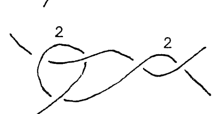
Only an outline of the methods used, proceeding largely by examples, can be given. The following diagrams show the first few *integer tangles* with their designations.

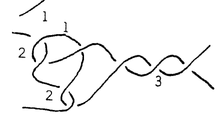


Integer tangles 1, 2, and 3

Integer tangles are combined to form *rational tangles*, as the following examples show:

21 

212 

2123 

Note that to form the tangle $abcd$ (where a, b, c, d represent integer tangles), first a is reflected in a leading diagonal then joined to b. Then the tangle ab is reflected and joined to c. Finally, abc is reflected and joined to d. The manner of joining two tangles is evident from the examples.

A tangle is turned into a knot by joining the two upper strings (loose ends), and then joining the two lower strings.

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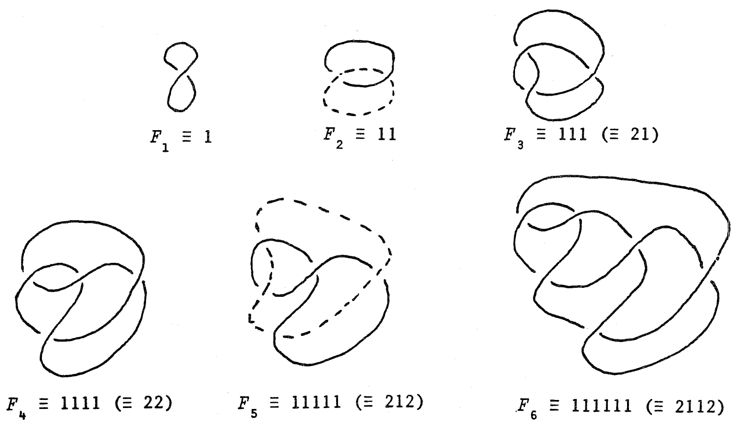
*As in [2], we use "knot" as an inclusive term for "μ-link," $\mu \geqslant 1$.

In [5] a table of diagrams of prime knots and links is given, showing the knots on n crossings, for $n = 2, 3, \ldots, 10$. Conway, in [2], classifies the knots and links through to $n = 11$ crossings.

2. THE FIBONACCI KNOT-CLASS

We now define what we have called the *Fibonacci knot-class* to be the sequence of rational knots which are, in Conway's constructional notation, 1, 11, 111, 1111, ... There is thus one knot in the class for each value of n-crossings; we give diagrams for the first six in the sequence before describing the properties that relate them to the Fibonacci numbers.



The Fibonacci knots to $n = 6$

In the sequence, each knot corresponds to its Fibonacci number through a certain knot-invariant to be described. Then when F_i is odd the knot is a 1-link, and when F_i is even the knot is a 2-link (where $\{F_i\}$ is the sequence 1, 2, 3, 5, 8, ...).

3. PROPERTIES OF THE FIBONACCI KNOT-CLASS

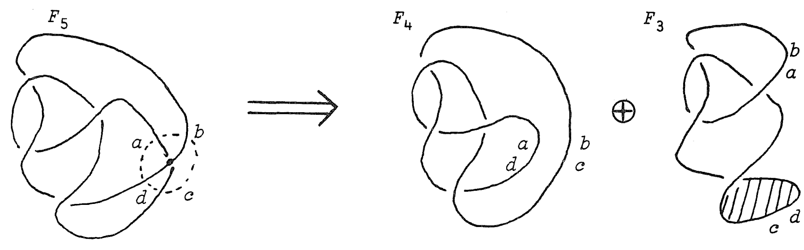
A Vertex-Deletion Operation; Production of "Twins"

If a crossing of a knot diagram is "cut-out" or "deleted," the four cut-ends may be joined again in two ways that lead to a pair of alternating knots, each having one fewer crossing than the original knot. We may call the original knot K, and the associated pair of knots which are obtainable from the vertex-deletion K' and K''; we may speak of K as the *parent* knot, and call (K', K'') a pair of *twins*.

Let us write, formally, that $K = K' \oplus K''$ whenever (K', K'') are twins from parent knot K.

Twins from the Fibonacci Knots

Consider, for example, the Fibonacci knot $F_5 \equiv 11111$. By its construction, the last 1 corresponds to the crossing on the far right of its diagram. We demonstrate that deletion of this vertex leads to the twins (F_4, F_3). Thus:



Cut out vertex

Join a to d and b to c

Join a to b and c to d

The knot on the far right is immediately seen to be equivalent to F_3 once the loop (shown shaded) is removed by twisting it once, out of the plane and back, through 180° clockwise.

To transform the first right-hand knot to the one shown in Section 2 requires two operations: (1) turn the entire knot over in the plane, rotating it about an axis in the plane that runs from NW to SE; (2) rotate the entire knot through 180° in the plane (about an axis perpendicular to the plane).

Similarly, we can show that, if we delete its last vertex, F_6 has twins (F_5, F_4), F_7 has twins (F_6, F_5), and so on. Using the symbol $\oplus$ as described above, we can write, formally,

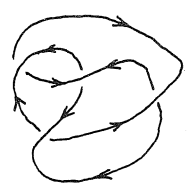
$$F_{n+2} = F_{n+1} \oplus F_n, \qquad n = 1, 2, \ldots,$$

which is the recurrence relation for the Fibonacci series.

The "Tree Number" Knot Invariant

The edges of an alternating knot-graph may be given orientations in such a way that the arrows alternate in direction as the knot is toured from edge to edge. We call this a *balanced alternating orientation.*

For a knot-graph with a balanced alternating orientation, we may count the number of directed spanning trees that emanate from any given vertex. We can show that this number is independent of the vertex chosen as root and, further, that it is a knot-invariant for alternating knots. The first proof of invariance of this *tree number* (τ) may be found in [3].



$\tau = 5$ (whichever vertex is taken as root; and whichever alternating diagram is used to represent the knot).

Example: Knot F_4, with balanced alternating orientation

Computation of τ for the Rational Knots

In [6] we derive the following recurrence equations for

$\tau(m_1 m_2 \ldots m_c)$,

the tree number of the rational knot $m_1 m_2 \ldots m_c$.

$\tau(m_1) = m_1$

$\tau(m_1 m_2) = m_2 m_1 + 1$

....

$\tau(m_1 m_2 \ldots m_c) = m_c \cdot \tau(m_1 m_2 \ldots m_{c-1}) + \tau(m_1 m_2 \ldots m_{c-2}).$

The tree numbers of the Fibonacci knot-class are given by setting $m_i = 1$, $i = 1, \ldots, c$. This gives

$$\tau(F_1) = 1, \quad \tau(F_2) = 2, \ \ldots, \ \tau(F_i) = \tau(F_{i-1}) + \tau(F_{i-2}).$$

Therefore, in this knot-class the tree numbers follow the Fibonacci sequence.

Consider the rational knot $m_1 m_2 \ldots m_c$, and the associated *continued fraction (terminated)* (C.F.):

$$\text{C.F.}(m_1 m_2 \ldots m_c) \equiv m_c + \frac{1}{m_{c-1}} + \frac{1}{m_{c-2}} + \cdots + \frac{1}{m_1}.$$

In view of the recurrence equations, the following is true:

$$\text{C.F.}(m_1 m_2 \ldots m_c) = \frac{\tau(m_1 m_2 \ldots m_c)}{\tau(m_1 m_2 \ldots m_{c-1})}.$$

This gives the following formula for the tree number of the c^{th} Fibonacci knot:

$$\tau(F_c) = \sum_{i=1}^{c} \text{C.F.}(F_i).$$

It should be noted here that Conway derives some interesting topological properties relating to the continued fraction of a rational knot in [2].

Other Interpretations of the Number τ

There are a number of knot invariants which have the same value as τ for any given knot. We list three here; a fuller discussion of them can be found in [6].

Entities equal in value to τ

(1) The torsion number of the two-fold branched cyclic covering space of the knot [1].

(2) The number of Euler circuits on the knot-digraph [4].

(3) The quantity $|\Delta(-1)|$, where $\Delta(x)$ is the Alexander polynomial of the knot [5].

Thus, for the Fibonacci knots, all of these invariant values follow the Fibonacci sequence.

On Parity of Tree Numbers

In [6], we show that τ is odd if and only if the knot-graph is a 1-link (i.e., one string). In the Fibonacci knot sequence, then, the knots F_1, F_3,

F_4, F_6, F_7, ... are 1-links; it is easy to show that every third knot, with even τ, is a 2-link. That is F_2, F_5, F_8, ... are 2-links.

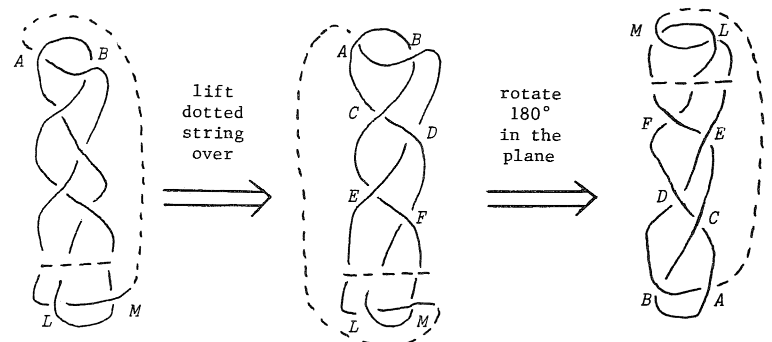
On Amphicheirality

A knot is *amphicheiral* if it can be transformed into its mirror image by a bi-continuous transformation (that is, without cutting and rejoining the string).

In Conway's notation, the mirror image of 1 1 ... 1 is $\bar{1}\,\bar{1}\ldots\bar{1}$; the symbol $\bar{1}$ denotes a crossing $\diagup\!\!\!\diagdown$.

Proposition: F_c is amphicheiral for c = 1, 2, 4, 6, ... (c even after 1).

Proof: For c = 1 and 2, it is easy to note how the transformation can be carried out. For general c, the necessary transformations to carry the knot into its mirror image are as follows:



Knot: 111111...11 Knot: $\bar{1}\bar{1}\ldots\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}$

It is well known that knots with an odd number of crossings cannot be amphicheiral; hence, F_i, where i = 3, 5, ... are not amphicheiral.

4. GENERALIZATIONS

An obvious generalization of the above work would be to study the *knot-classes* $\{F_i^{(m)}\}$, where

$\{F_i^{(1)}\} \equiv \{F_i\}$ is the Fibonacci class,

$\{F_i^{(2)}\}$ is the class of rational knots 2, 22, 222, 2222, ...,

$\{F_i^{(3)}\}$ is the class 3, 33, 333, 3333, ...,

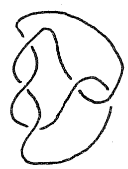
etc.

Knots with i = 2, 4, ... (even) in each sequence are amphicheiral.

The *tree numbers* of knots in these classes satisfy the equations of Section 3. For $m = 1$, they are the Fibonacci numbers; for $m = 2$, the Pell numbers.

Doubtless the properties of these numbers, which form interesting two-way sequences, are well known.

Any rational knot may be represented as a formal sum of knots of type $F_i^{(m)}$, making use of the vertex deletion operation described in Section 3. Such representations are not in general unique (that is, a given knot may have more than one representation), but it is conjectured that any representation is an invariant of that knot. For example, the knot (32) shown below may be represented in the following ways, by various vertex deletions:



$$\Longrightarrow \textit{knot}\ (31) \oplus F_3 \Rightarrow F_3 \oplus F_1 \oplus F_3 \Rightarrow 2F_3 \oplus F_1$$

$$\overset{\text{also}}{\Longrightarrow} \quad F_4 \oplus F_2$$

Knot (32)

Note that to each representation there corresponds a linear decomposition of the knot's tree number into Fibonacci numbers; e.g., for the knot (32) we have $\tau = 7$, with the corresponding decompositions $5 + 2$ and $2 \times 3 + 1$.

It would be exciting if a study of number sequences associated with knot-classes were to lead to methods for counting more general classes of knots. There are virtually no results in this area, to my knowledge.

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