ON SOME POLYGONAL NUMBERS WHICH ARE, AT THE SAME TIME, THE SUMS, DIFFERENCES, AND PRODUCTS OF TWO OTHER POLYGONAL NUMBERS

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We denote the n^{th} g-gonal number by

$$P_{n,g} = n\{(g - 2)n - (g - 4)\}/2.$$

For $g = 3, 5, 6$, and 8, we denote $P_{n,g}$ by T_n, the triangular numbers, P'_n, the pentagonal numbers, H_n, the hexagonal numbers, and O_n, the octagonal numbers, respectively. We denote $P_{n,g}$ by P_n whenever there is no danger of confusion.

Sierpiński [18] has proved that "there exist an infinite number of triangular numbers which are, at the same time, the sums, differences and products of two other triangular numbers > 1." Ando [1] proved that "there exist an infinite number of g-gonal numbers that can be expressed as the sum and difference of two other g-gonal numbers at the same time." It was also shown in [6] that there are an infinite number of g-gonal numbers that can be expressed as the product of two other g-gonal numbers.

The present paper will show that there are infinitely many g-gonal numbers ($g = 5, 6$, and 8) which are at the same time the sums, differences, and products of two other g-gonal numbers.

1. THE EQUATION $P_{u+w} + P_{v+w} = P_{u+v+w}$

If $P_x + P_y = P_z$, by putting $u = z - y$, $v = z - x$, and $w = x + y - z$, we have $x = u + w$, $y = v + w$, and $z = u + v + w$. However, a little algebra shows that $P_{u+w} + P_{v+w} = P_{u+v+w}$ implies $2(g - 2)uv = (g - 2)w(w - 1) + 2w$. Hence

Theorem 1: Any solution x, y, z of the equation $P_x + P_y = P_z$ can be expressed as $x = u + w$, $y = v + w$, $z = u + v + w$, where

$$w \equiv 0 \pmod{g - 2}$$

and

$$uv = \{(g - 2)w^2 - (g - 4)w\}/2(g - 2).$$

Using this theorem, which is a generalization of the work of Fauquembergue [7] and of Shah [15] on triangular numbers, we can obtain the solutions of the equation $P_x + P_y = P_z$ in an efficient way. For example, we have the following table for $g = 5$.

Table 1. $P'_x + P'_y = P'_z$ $(w \leqslant 9,\ u \leqslant v)$

w	$(3w^2 - w)/6$	u	v	x	y	z
3	4	2	2	5	5	7
		1	4	4	7	8
6	17	1	17	7	23	24
9	39	3	13	12	22	25
		1	39	10	48	49

If we put $v + w = w'$ in $P_{1+v+w} = P_{1+w} + P_{v+w}$ and $P_{1+w'} = P_{1+v'+w'} - P_{v'+w'}$, then we obtain g-gonal numbers that can be expressed as the sum and difference of two other g-gonal numbers at the same time.

Corollary: If $w \equiv 0 \pmod{(g-2)^2}$ and $v = \{(g-2)w^2 - (g-4)w\}/2(g-2)$, then we have

$$P_{v+w+1} = P_{w+1} + P_{v+w} = P_a - P_b, \text{ where}$$

$$a = \{(g-2)(v+w)^2 - (g-4)(v+w)\}/2(g-2) + v + w + 1$$

and

$$b = \{(g-2)(v+w)^2 - (g-4)(v+w)\}/2(g-2) + v + w.$$

Putting $w = x - 1$ for $g = 3$, we obtain a result of Sierpiński [18]; putting $w = 9n$ for $g = 5$, $w = 16n$ for $g = 6$, $w = 25k$ for $g = 7$, and $w = 36n$ for $g = 8$, we obtain the results of Hansen [9], O'Donnell [13], Hindin [10], and O'Donnell [14], respectively.

2. THE EQUATION $P_{at-d} + P_{bt-e} = P_{ct-f}$

In this section we study somewhat more general second-degree sequences than P_n, and obtain necessary and sufficient conditions for certain infinite families of representations to exist. We then specialize to polygonal numbers. To this end, let $F(\alpha, \beta; n) = n(\alpha n - \beta)$, where α, β are integers with $(\alpha, \beta) = 1$ and $\alpha > 0$.

Theorem 2: Let a, b, c, d, e, and f be integers with a, b, and c positive and $(a, b, c) = 1$. A necessary and sufficient condition for the identity in t,

$$F(\alpha, \beta; at - d) + F(\alpha, \beta; bt - e) = F(\alpha, \beta; ct - f),$$

to hold is that there exist integers p, q, r, and s that satisfy equations (0) and (I), or (0) and (II):

$$\begin{cases} a = (p+q)(p-q),\ b = 2pq,\ c = p^2 + q^2, \\ (p, q) = 1,\ p > q > 0,\ p + q \equiv 1 \pmod 2, \end{cases} \tag{0}$$

$$\begin{cases} d = (p+q)r, \ e = \dfrac{q}{\alpha}s, \ f = (p-q)r + \dfrac{q}{\alpha}s, \\ q \equiv 0 \pmod{\alpha}, \ 2\alpha pr - (p-q)s = -\beta, \end{cases} \tag{I}$$

$$\begin{cases} d = \dfrac{p-q}{\alpha}r, \ e = ps, \ f = \dfrac{p-q}{\alpha}r + qs, \\ p \equiv q \pmod{\alpha}, \ 2qr - \alpha(p+q)s = \beta. \end{cases} \tag{II}$$

Proof: In order for the desired identity in t,

$$(at-d)(\alpha at - \alpha d - \beta) + (bt-e)(\alpha bt - \alpha e - \beta) = (ct-f)(\alpha ct - \alpha f - \beta),$$

to hold, it is necessary and sufficient that the equations

$$a^2 + b^2 = c^2, \tag{1}$$

$$(2\alpha d + \beta)a + (2\alpha e + \beta)b = (2\alpha f + \beta)c, \tag{2}$$

$$(\alpha d + \beta)d + (\alpha e + \beta)e = (\alpha f + \beta)f \tag{3}$$

be valid.

From (2),

$$cf = ad + be + \frac{\beta(a+b-c)}{2\alpha}, \tag{4}$$

and from (1), (3), and (4), we obtain

$$\begin{aligned} (a^2+b^2)\{(\alpha d+\beta)d + (\alpha e + \beta)e\} &= c^2(\alpha f + \beta)f \\ &= \alpha(cf)^2 + \beta c(cf) \\ &= \alpha\left\{ad + be + \frac{\beta(a+b-c)}{2\alpha}\right\}^2 + \beta c\left\{ad + be + \frac{\beta(a+b-c)}{2\alpha}\right\}. \end{aligned}$$

Expanding and transforming the above, we have

$$\alpha(bd-ae)^2 - \beta(a-b)(bd-ae) - \frac{\beta^2 ab}{2\alpha} = 0.$$

Hence,

$$\begin{cases} \text{(a)} \quad bd - ae = \dfrac{\beta(a-b-c)}{2\alpha}, \text{ or} \\ \text{(b)} \quad bd - ae = \dfrac{\beta(a-b+c)}{2\alpha}. \end{cases} \tag{5}$$

Now, for positive integers a, b, and c with $(a, b, c) = 1$ and b even, the solutions of (1) are given by

$$\text{(0)} \quad \begin{cases} a = (p+q)(p-q), \ b = 2pq, \ c = p^2 + q^2, \text{ where} \\ p \text{ and } q \text{ are positive integral parameters with} \\ (p, q) = 1, \ p > q > 0, \text{ and } p + q \equiv 1 \pmod{2}. \end{cases}$$

Equations (6) and (7) below are necessary for (4) and (5) to hold.

$$\beta(a+b-c) = 2\beta q(p-q) \equiv 0 \pmod{2\alpha}, \tag{6}$$

$$\begin{cases} \text{(a)} \quad \beta(a - b - c) = -2\beta q(p + q) \equiv 0 \pmod{2\alpha}, \text{ or} \\ \text{(b)} \quad \beta(a - b + c) = 2\beta p(p - q) \equiv 0 \pmod{2\alpha}. \end{cases} \tag{7}$$

Since $(\alpha, \beta) = 1$ and $(p, q) = 1$, (6) and (7) hold only if

$$\begin{cases} \text{(a)} \quad q \equiv 0 \pmod{\alpha}, \text{ or} \\ \text{(b)} \quad p \equiv q \pmod{\alpha}. \end{cases} \tag{8}$$

(I) If $q \equiv 0 \pmod{\alpha}$, (5)(a) becomes

$$2pqd - (p + q)(p - q)e = -\beta\frac{q}{\alpha}(p + q),$$

so that we have

$$2\alpha pr - (p - q)s = -\beta, \text{ where } d = (p + q)r \text{ and } e = \frac{q}{\alpha}s.$$

Substituting this into (4), we have $f = (p - q)r + \frac{q}{\alpha}s$.

(II) If $p \equiv q \pmod{\alpha}$, (5)(b) becomes

$$2pqd - (p + q)(p - q)e = \beta p \cdot \frac{p - q}{\alpha},$$

so that we have

$$2qr - \alpha(p + q)s = \beta, \text{ where } d = \frac{p - q}{\alpha}r \text{ and } e = ps.$$

Substituting this into (4), we have $f = \frac{p - q}{\alpha}r + qs$. Thus, we have the equivalence relation

$$(1) \cdot (2) \cdot (3) \Leftrightarrow (0) \cdot (4) \cdot (5) \Leftrightarrow (0) \cdot (\text{I}) \text{ or } (0) \cdot (\text{II}),$$

which proves Theorem 2.

Corollary: Solutions of $P_x + P_y = P_z$ are obtained by $x = at - d$, $y = bt - e$, $z = ct - f$. We use Theorem 2 by putting

$$P_{n,g} = \frac{1}{2}F(g - 2, g - 4; n) \text{ for } g \text{ odd, and}$$

$$P_{n,g} = F\left(\frac{g - 2}{2}, \frac{g - 4}{2}; n\right) \text{ for } g\ (\neq 4) \text{ even.}$$

In the case $g = 4$, we obtain a, b, and c from Theorem 2 (0) by putting $d = e = f = 0$.

Example: If $g = 5$, then $\alpha = 3$, $\beta = 1$. Since $q \equiv 0 \pmod 3$, or $p \equiv q \pmod 3$, and $(p, q) = 1$, $p > q > 0$, $p + q \equiv 1 \pmod 2$, we have

$$q = 1;\ p = 4, 10, 16, \ldots,$$

$$q = 2;\ p = 5, 11, 17, \ldots,$$

$$q = 3;\ p = 4, 8, 10, 14, 16, \ldots .$$

When $p = 4$, $q = 1$, $2qr - \alpha(p+q)s = \beta$ becomes $2r - 15s = 1$, where one solution is $r = 8$, $s = 1$. Using these values in (0) • (II), we obtain

$$a = 15,\ b = 8,\ c = 17,\ d = 8,\ e = 4,\ f = 9,$$

and

$$P'_{15t-8} + P'_{8t-4} = P'_{17t-9}.$$

Changing t into $8t - 3$ and $17t - 7$, we have

$$P'_{136t-60} = P'_{120t-53} + P'_{64t-28} = P'_{289t-128} - P'_{255t-113}.$$

Table 2. $P'_{at-d} + P'_{bt-e} = P'_{ct-f}$, $P'_x + P'_y = P'_z$ ($z \leqslant 30$)

	p	q	r	s	a	b	c	d	e	f	t	x	y	z
(II)	4	1	8	1	15	8	17	8	4	9	1	7	4	8
											2	22	12	25
(I)	4	3	0	1	7	24	25	0	1	1	1	7	23	24
(II)	5	2	16	3	21	20	29	16	15	22	1	5	5	7
(I)	8	3	3	29	55	48	73	33	29	44	1	22	19	29

Table 3. Correspondence of the Solutions of $P_x + P_y = P_z$ in [1] Ex. 1

g	Parity	Case	p	q	r	s	t
k:even		(I)	$\frac{(k-2)^2}{2}t+1$	$\frac{(k-2)^2}{2}t$	0	$\frac{k-4}{2}$	1
k:odd	t:even	(I)	$\frac{(k-2)^2}{2}t+1$	$\frac{(k-2)^2}{2}t$	0	$k-4$	1
	t:odd	(II)	$(k-2)^2t+1$	1	$\frac{(k-2)^3t+(3k-8)}{2}$	1	1

3. THE EQUATIONS $P_z = P_x + P_y = P_u - P_v = P_rP_s$

For $g \neq 4$, if $(g - 2)P_n - (g - 4) = 2P_m$, we conjecture that $P_{P_n} = P_nP_m$ can be expressed as the sum and difference of two other g-gonal numbers. But we cannot prove this. However, we have

Theorem 3: There exist an infinite number of hexagonal numbers that can be expressed as the sum-difference-product of two other hexagonal numbers.

Proof: If we assume $H_n = H_3H_m$, then we have $(4n - 1)^2 - 15(4m - 1)^2 = -14$. By putting $N = 4n - 1$, $M = 4m - 1$, we get $N^2 - 15M^2 = -14$. Its complete solution is given by the formulas

(i) $N_i + \sqrt{15}M_i = \pm(1 + \sqrt{15})(4 + \sqrt{15})^i$

and

(ii) $N_i + \sqrt{15}M_i = \pm(-1 + \sqrt{15})(4 + \sqrt{15})^i$,

where $i = 0, \pm 1, \pm 2, \pm 3, \ldots$.

In (i), if $N_i + \sqrt{15}M_i > 0$, $i > 0$, and $i \equiv 2 \pmod 4$, then $N_i \equiv M_i \equiv -1 \pmod 4$. N_i satisfies a recurrence relation

$$N_{i+2} = 8N_{i+1} - N_i,$$

which leads to $N_{i+4} = 62N_{i+2} - N_i$. Also, by repetition, $N_{i+8} = 3842N_{i+4} - N_i$. From $4n_{i+8} - 1 = 3842(4n_{i+4} - 1) - (4n_i - 1)$, it follows that $n_{i+8} = 3842n_{i+4} - n_i - 960$. Changing $4i - 2$ into i, it becomes

$$n_{i+2} = 3842n_{i+1} - n_i - 960,$$

with initial values $n_1 = 38$, $n_2 = 145058$. Similarly, we get

$$m_{i+2} = 3842m_{i+1} - m_i - 960,$$

with initial values $m_1 = 10$, $m_2 = 37454$.

For all i, we have

$$\begin{aligned} H_{n_i} = H_3 H_{m_i} &= 15m_i(2m_i - 1) \\ &= (4m_i - 1)(8m \;\; - 3) - (m_i - 1)(2m_i - 3) \\ &= H_{4m_i - 1} - H_{m_i - 1}. \end{aligned}$$

For $i \equiv 1 \pmod 7$, we have $n_i \equiv -1 \pmod{13}$. On taking $t = (n_i + 1)/13$ in

$$H_{13t-1} = H_{5t} + H_{12t-1},$$

we get

$$H_{n_i} = H_{(5n_i+5)/13} + H_{(12n_i-1)/13}.$$

Thus, for $i \equiv 1 \pmod 7$, H_{n_i} is expressed as the sum-difference-product of two other hexagonal numbers. If we put $i = 1$, then we have

$$H_{38} = H_{15} + H_{35} = H_{39} - H_9 = H_3 H_{10}.$$

In a similar way, we obtain

Theorem 4: For $g = 5$ and 8, there exist an infinite number of g-gonal numbers that can be expressed as the sum-difference-product of two other g-gonal numbers.

Proof: If we put

$$\left.\begin{aligned} n_1 = 4,\ n_2 = 600912,\ n_{i+2} &= 155234n_{i+1} - n_i - 25872 \\ m_1 = 1,\ m_2 = 128115,\ m_{i+2} &= 155234m_{i+1} - m_i - 25872 \end{aligned}\right\} \quad i = 1, 2, 3, \ldots,$$

then, for $i \equiv 9 \pmod{14}$, we have $n_i \equiv 7 \pmod{29}$ and $m_i \equiv 1 \pmod 2$, so that

$$P'_{n_i} = P'_{(21n_i - 2)/29} + P'_{(20n_i + 5)/29} = P'_{(23m_i - 7)/2} - P'_{(21m_i - 7)/2}$$
$$= P'_4 P'_{m_i}.$$

Also, if we put

$$\left.\begin{array}{l} n_1 = 304,\ n_2 = 1345421055984, \\ n_{i+2} = 4430499842n_{i+1} - n_i - 1476833280 \\ m_1 = 38,\ m_2 = 166878943590, \\ m_{i+2} = 4430499842m_{i+1} - m_i - 1476833280 \end{array}\right\} \quad i = 1, 2, 3, \ldots,$$

then, for $i \equiv 0, 1 \pmod 7$, we have $n \equiv 14 \pmod{29}$, so that

$$O_{n_i} = O_{(21n_i - 4)/29} + O_{(20n_i + 10)/29} = O_{9m_i - 4} - O_{4m_i - 4} = O_5 O_{m_i}.$$

Here, if we put $i = 1$, then we have

$$O_{304} = O_{220} + O_{210} = O_{338} - O_{148} = O_5 O_{38}.$$

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LETTER TO THE EDITOR

19 December 1985

Dear Editor:

Before the publication of my article, "Generators of Unitary Amicable Numbers," in the May 1985 issue of The Fibonacci Quarterly, Dr. H. J. J. te Riele and I exchanged letters concerning unitary amicable numbers. He pointed out that his report, NW 2/78, published by the Matematisch Centrum in Amsterdam (with which he is affiliated), contains many of the results in my paper, albeit from a slightly different point of view. Both references to these letters and to report NW 2/78 were inadvertently omitted from my article.

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Sincerely yours,

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