# EXPLICIT FORMULAS FOR NUMBERS OF RAMANUJAN

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## 1. INTRODUCTION

In Chapter 3 of his second notebook [1, p. 165], Ramanujan defined numbers $a(n, k)$ such that $a(2, 0) = 1$ and for $n \geqslant 2$,

$$a(n + 1, k) = (n - 1)a(n, k - 1) + (2n - 1 - k)a(n, k). \tag{1.1}$$

He defined $a(n, k) = 0$ when $k < 0$ or $k > n - 2$. The numbers were used in the following way: Fix $a > 1/e$ and for real $h$ define $x > 0$ by the relation

$$x^x = a^a e^h.$$

Then it can be shown [1, pp. 164-165] that

$$\frac{x - a}{a} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A_n}{n!}\left(\frac{h}{a}\right)^n,$$

where $|h|$ is sufficiently small, $A_1 = (1 + \log a)^{-1}$,

$$A_n = \sum_{k=0}^{n-2} a(n, k)(1 + \log a)^{1+k-2n}, \quad n \geqslant 2.$$

The values of $a(n, k)$ for $2 \leqslant n \leqslant 7$ are given in the following table.

Table 1

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 2 | 1 | | | | | |
| 3 | 3 | 1 | | | | |
| 4 | 15 | 10 | 2 | | | |
| 5 | 105 | 105 | 40 | 6 | | |
| 6 | 945 | 1260 | 700 | 196 | 24 | |
| 7 | 10395 | 17325 | 12600 | 5068 | 1148 | 120 |

The purpose of this paper is to show how $a(n, k)$ can be expressed in terms of Stirling numbers of the first kind and associated Stirling numbers of the second kind. We prove in §2 that

$$a(n, n - 2) = (n - 2)! = (-1)^n s(n - 1, 1),$$
$$a(n, n - 3) = (-1)^n (n - 2)s(n - 1, 1) + (-1)^{n-1} 2s(n - 1, 2),$$

and in general, for $k \geqslant 2$,

$$a(n,\ n - k) = \sum_{t=1}^{k-1} (-1)^{n-1-t} P_{k,\,t}(n) s(n - 1,\ t),$$

where $P_{k,\,t}(n)$ is a polynomial in $n$ of degree $k - 1 - t$ and $s(n - 1,\ t)$ is the Stirling number of the first kind. A recurrence formula for the coefficients of $P_{k,\,t}(n)$ is derived and the values of $P_{k,\,t}(n)$ for $2 \leqslant k \leqslant 6$ are computed (see Table 2). In §3 we show that

$$a(n,\ 0) = b(2n - 2,\ n - 1),\quad a(n,\ 1) = b(2n - 3,\ n - 2),$$

and, for $k > 1$,

$$a(n,\ k) = \sum_{r=3}^{n-k+1} Q_k(n,\ r)\frac{(2n - k - 3)!!}{(2r - 3)!!}(r - 1)b(2r - 3,\ r - 2),$$

where the $Q_k(n,\ r)$ are rational numbers,

$$n!! = \begin{cases} 1.3 \cdots n & \text{if } n \text{ is odd,} \\ 2.4 \cdots n & \text{if } n \text{ is even,} \end{cases} \tag{1.2}$$

and $b(n,\ k)$ is the associated Stirling number of the second kind. A recurrence formula for $Q_k(n,\ r)$ is worked out and the values of $Q_k(n,\ r)$ for $k = 2$ and $k = 3$ are given. In §4 we prove an identity for the Stirling numbers of the first kind. This identity, interesting in its own right, is used in the proof of Theorem 2.1.

## 2. STIRLING NUMBERS OF THE FIRST KIND

Throughout the paper we use the notation

$$(x)_n = x(x - 1) \cdots (x - n + 1).$$

The Stirling number of the first kind, $s(n,\ k)$, can be defined by means of

$$(x)_n = \sum_{k=0}^{n} s(n,\ k)x^k. \tag{2.1}$$

These numbers are well known and have been extensively studied; a table of values for $1 \leqslant n \leqslant 15$ can be found in [2, p. 310]. In particular,

$$s(n,\ 1) = (-1)^{n-1}(n - 1)!.$$

By (1.1) and the fact that $a(n,\ k) = 0$ for $k > n - 2$, we have

$$a(n,\ n - 2) = (n - 2)a(n - 1,\ n - 3) = (n - 2)!a(2,\ 0) = (n - 2)!,$$

and therefore

$$a(n,\ n - 2) = (-1)^n s(n - 1,\ 1).$$

**Theorem 2.1:** For $k \geqslant 2$,

$$a(n,\ n - k) = \sum_{t=1}^{k-1} P_{k,\,t}(n)(-1)^{n-1-t} s(n - 1,\ t),$$

where $P_{k,t}(n)$ is a polynomial in $n$ of degree $k - 1 - t$. The coefficient of $n^{k-1-t}$ is $t!/(k - t - 1)!$. If we write

$$P_{k,t}(n) = \sum_{j=0}^{k-1-t} c_k(t, j)(n - 1)_j = \sum_{j=0}^{k-1-t} d_k(t, j)(n - 2)_j,$$

then, for $k > 2$, $d_k(1, 0) = 0$, $d_k(t, 0) = (k - 1)c_{k-1}(t - 1, 0)$ for $t > 1$, and

$$d_k(t, j) = \sum_{m=t}^{k-1-j} (-1)^{m-t}\left(\frac{1}{j}\right)^{m-t+1}(d_{k-1}(m, j - 1) + (k - 1)c_{k-1}(m, j))$$

for $t \geqslant 1$, $j > 0$.

**Proof**: We showed above that the theorem is true for $a(n, n - 2)$; assume it is true for $a(n, n - (k - 1))$, so we can write

$$a(n, n - (k - 1)) = \sum_{m=1}^{k-2} P_{k-1,m}(n)(-1)^{n-1-m}s(n - 1, m), \tag{2.2}$$

$$P_{k-1,m}(n) = \sum_{j=0}^{k-2-m} c_{k-1}(m, j)(n - 1)_j = \sum_{j=0}^{k-2-m} d_{k-1}(m, j)(n - 2)_j. \tag{2.3}$$

By (1.1), we have the recurrence

$$a(n, n - k) = (n - 2)a(n - 1, n - 1 - k) + (n - 3 + k)a(n - 1, n - k). \tag{2.4}$$

We define the formal power series

$$A_k(x) = \sum_{n=k}^{\infty} a(n, n - k)\frac{x^{n-1}}{(n - 2)!}$$

and sum on both sides of (2.4), after multiplying by $\frac{x^{n-1}}{(n - 2)!}$, to obtain

$$A_k(x) = xA_k(x) + \sum_{n=k}^{\infty} (n - 3 + k)a(n - 1, n - k)\frac{x^{n-1}}{(n - 2)!}.$$

Therefore,

$$A_k(x) = \frac{1}{1 - x} \sum_{n=k-1}^{\infty} (n - 2 + k)a(n, n - (k - 1))\frac{x^n}{(n - 1)!}. \tag{2.5}$$

Comparing coefficients of $x^{n-1}$ in (2.5), we have

$$a(n, n - k) = \sum_{r=k-1}^{n-1} \frac{(n - 2)!}{(r - 2)!}\, a(r, r - (k - 1))$$
$$+ (k - 1)\sum_{r=k-1}^{n-1} \frac{(n - 2)!}{(r - 1)!}\, a(r, r - (k - 1)). \tag{2.6}$$

We now substitute into (2.6) the formula for $a(r, r - (k - 1))$ given by (2.2) and (2.3). Then (2.6) becomes, after some manipulation,

$$a(n, n-k) = \sum_{m=1}^{k-2} \sum_{j=0}^{k-2-m} (-1)^{m-n} d_{k-1}(m, j) \sum_{r=j+1}^{n-2} \frac{(n-2)!(-1)^{n-r}}{(r-j-1)!} s(r, m)$$

$$+ (k-1) \sum_{m=1}^{k-2} \sum_{j=0}^{k-2-m} (-1)^{m-n} c_{k-1}(m, j) \sum_{r=j}^{n-2} \frac{(n-2)!(-1)^{n-r}}{(r-j)!} s(r, m). \quad (2.7)$$

At this point we need the following lemma, which we prove in §4.

**Lemma 2.1:** We have

$$\sum_{r=j}^{n} \frac{n!(-1)^{n-r}}{(r-j)!} s(r, m) = \begin{cases} -(n)_j \sum_{t=1}^{m} s(n+1, t)\left(\frac{1}{j}\right)^{m-t+1} & \text{if } j > 0, \\ s(n+1, m+1) & \text{if } j = 0. \end{cases}$$

We now substitute the formulas of Lemma 2.1 (with $n$ replaced by $n-2$ and $j$ replaced by $j+1$) into (2.7) and change the order of the $m$, $t$ summations. We have

$$\begin{aligned} a(n, n-k) &= \sum_{t=1}^{k-2} \sum_{j=0}^{k-1-t} d_k(t, j)(n-2)_j (-1)^{n-1-t} s(n-1, t) \\ &\quad + d_k(k-1, 0)(-1)^{n-k} s(n-1, k-1) \\ &= \sum_{t=1}^{k-1} P_{k,t}(n)(-1)^{n-1-t} s(n-1, t) \end{aligned}$$

where $d_k(1, 0) = 0$, $d_k(t, 0) = (k-1)c_{k-1}(t-1, 0)$ for $t > 1$ and

$$d_k(t, j) = \sum_{m=t}^{k-1-j} (-1)^{m-t}\left(\frac{1}{j}\right)^{m-t+1} (d_{k-1}(m, j-1) + (k-1)c_{k-1}(m, j))$$

for $t \geqslant 1$, $j > 0$. It follows that $P_{k,t}(n)$ has degree $k-1-t$ and the coefficient of $n^{k-1-t}$ is

$$\begin{aligned} d_k(t, k-1-t) &= \frac{1}{k-1-t} d_{k-1}(t, k-2-t) \\ &= \frac{1}{(k-1-t)!} d_{t+1}(t, 0). \end{aligned}$$

Since

$$\begin{aligned} P_{k,k-1}(n) = d_k(k-1, 0) &= (k-1)c_{k-1}(k-2, 0) \\ &= (k-1)d_{k-1}(k-2, 0) = (k-1)!, \end{aligned}$$

the coefficient of $n^{k-1-t}$ in $P_{k,t}(n)$ is $\frac{t!}{(k-1-t)!}$. This completes the proof of Theorem 2.1.

From Theorem 2.1, we have the following special cases:

$$\begin{aligned} &c_k(t, j) = 0 = d_k(t, j) \quad \text{if } j > k-1-t, \\ &c_k(k-2, 1) = d_k(k-2, 1) = (k-2)! \\ &c_k(k-2, 0) = (k-1)! + (-1)^{k-1}s(k, 2), \\ &d_k(k-2, 0) = k(k-2)! + (-1)^{k-1}s(k, 2), \end{aligned}$$

$$c_k(t,\ k-2-t) = [t!(k-2) + (-1)^t(t+1)s(t+1,\ 2)]/(k-2-t)!,$$
$$d_k(t,\ k-2-t) = [t!(k-1) + (-1)^t(t+1)s(t+1,\ 2)]/(k-2-t)!.$$

It follows that

$$P_{k,\,k-2}(n) = (k-2)!n + (k-2)(k-2)! + (-1)^{k-1}s(k,\ 2). \tag{2.8}$$

We have already pointed out that

$$P_{k,\,k-1}(n) = (k-1)!. \tag{2.9}$$

The evidence seems to indicate that

$$P_{k,\,1}(n) = \binom{n+k-5}{k-2},$$

but this has not been proved.

Since $(n-1)_j = (n-2)_j + j(n-2)_{j-1}$, we have the relationship:

$$d_k(t,\ j) = c_k(t,\ j) + (j+1)c_k(t,\ j+1). \tag{2.10}$$

Since

$$(n-2)_j = \sum_{r=0}^{j}(-1)^{j-r}j!(n-1)_r/r!,$$

we have

$$c_k(t,\ j) = (-1)^j\sum_{r=j}^{k-1-t}(-1)^r r!d_k(t,\ r)/j!. \tag{2.11}$$

Using (2.10) and (2.11), we can obviously write the recurrence for the coefficients $P_{k,\,t}(n)$ in several different ways.

The following values of $P_{k,\,t}(n)$ have been worked out using Theorem 2.1.

Table 2

| $k \backslash t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 2 | 1 | | | | |
| 3 | $n-2$ | 2 | | | |
| 4 | $\binom{n-1}{2}$ | $2n-7$ | 6 | | |
| 5 | $\binom{n}{3}$ | $(n-2)(n-4)$ | $6n-32$ | 24 | |
| 6 | $\binom{n+1}{4}$ | $2\binom{n-1}{3} - \binom{n-1}{2}$ | $3n^2 - 29n + 61$ | $24n - 178$ | 120 |

## 3. ASSOCIATED STIRLING NUMBERS

The associated Stirling number of the second kind, $b(n,\ k)$, can be defined by means of

$$(e^x - x - 1)^k = k!\sum_{n=2k}^{\infty}b(n,\ k)\frac{x^n}{n!}.$$

We are using the notation of Riordan [3, pp. 74-78] for these numbers. They are also discussed in [2, pp. 221-222], where the notation $S_2(n, k)$ is used. A recurrence formula is

$$b(n + 1, k) = kb(n, k) + nb(n - 1, k - 1) \tag{3.1}$$

with $b(0, 0) = 1$ and $b(n, k) = 0$ if $n < 2k$. A table of values for $b(n, k)$, $1 \leqslant n \leqslant 18$, is given in [2, p. 222]. It follows from (3.1) that

$$b(2n, n) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1) = (2n - 1)!!,$$

with the notation of (1.2).

Since $a(n, 0) = (2n - 3)a(n - 1, 0) = (2n - 3)!!$, we have

$$a(n, 0) = b(2n - 2, n - 1), \quad n \geqslant 2. \tag{3.2}$$

Also,

$$\begin{aligned} b(2n - 1, n - 1) &= (n - 1)b(2n - 2, n - 1) + (2n - 2)b(2n - 3, n - 2) \\ &= (n - 1)a(n, 0) + (2n - 2)b(2n - 3, n - 2), \end{aligned} \tag{3.3}$$

with $b(3, 1) = 1$. Comparing (3.3) with (1.1), we have

$$a(n, 1) = b(2n - 3, n - 2), \quad n \geqslant 3. \tag{3.4}$$

Let $F_k(x)$ be the formal power series

$$\sum_{n=0}^{\infty} \frac{a(n + 1, k)}{(2n - k - 1)!!} x^n.$$

Then from (1.1) we have

$$F_k(x) = \frac{1}{1 - x} \sum_{n=0}^{\infty} \frac{(n - 1)a(n, k - 1)}{(2n - k - 1)!!} x^n. \tag{3.5}$$

Comparing coefficients of $x^{n-1}$ in (3.5), we have

$$a(n, k) = \sum_{j=k+1}^{n-1} \frac{(2n - k - 3)!!}{(2j - k - 1)!!}(j - 1)a(j, k - 1). \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$a(n, 2) = \sum_{r=3}^{n-1} \frac{(2n - 5)!!}{(2r - 3)!!}(r - 1)b(2r - 3, r - 2). \tag{3.7}$$

**Theorem 3.1:** For $k \geqslant 2$,

$$a(n, k) = \sum_{r=3}^{n-k+1} Q_k(n, r) \frac{(2n - k - 3)!!}{(2r - 3)!!}(r - 1)b(2r - 3, r - 2),$$

where the $Q_k(n, r)$ are rational numbers such that $Q_2(n, r) = 1$ and

$$Q_k(n, r) = \sum_{m=r+k-2}^{n-1} \frac{(2m - k - 2)!!}{(2m - k - 1)!!}(m - 1)Q_{k-1}(m, r)$$

for $3 \leqslant r \leqslant n - 1$ and $n \geqslant 4$.

**Proof**: According to (3.7), the Theorem is true for $a(n, 2)$; assume it is true for $a(n, k - 1)$. The proof for $a(n, k)$ follows immediately when we substitute

$$a(j, k - 1) = \sum_{r=3}^{j-k+2} Q_{k-1}(j, r)\frac{(2j - k - 2)!!}{(2r - 3)!!}(r - 1)b(2r - 3, r - 2)$$

into (3.6) and change the order of the summations. This completes the proof.

It is not difficult to evaluate

$$\begin{aligned} Q_3(n, r) &= \sum_{m=r+1}^{n-1} \frac{(2m - 5)!!}{(2m - 4)!!}(m - 1) \\ &= \frac{n(2n - 5)}{3 \cdot 4^{n-3}}\binom{2n - 6}{n - 3} - \frac{(r + 1)(2r - 3)}{3 \cdot 4^{r-2}}\binom{2r - 4}{r - 2}, \end{aligned}$$

but apparently the formulas for $Q_k(n, r)$ for $k > 3$ are complicated.

## 4. PROOF OF LEMMA 2.1

The second equality in Lemma 2.1 is proved in [2, p. 215]. To the writer's knowledge, the first equality is new and is of interest in its own right. We shall make use of the generating function

$$(1 + t)^u = \sum_{n=0}^{\infty} \sum_{k=1}^{n} s(n, k)u^k \frac{t^n}{n!}, \tag{4.1}$$

which follows from (2.1) and the MacLaurin series for $(1 + t)^u$. We have

$$t^j(u)_j (1 + t)^{u-j} = \sum_{n=j}^{\infty} \sum_{k=1}^{n} s(n, k)u^k \frac{t^n}{(n - j)!}, \tag{4.2}$$

so

$$t^j(u)_j (1 + t)^{u-j-1} = \sum_{n=j}^{\infty} \sum_{r=j}^{n} \left( \sum_{k=1}^{r} \frac{(-1)^{n-r}s(r, k)u^k}{(r - j)!} \right) \frac{t^n}{n!}. \tag{4.3}$$

From (4.1) and the binomial theorem,

$$(1 + t)^{u-j-1} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} s(n, k) \sum_{r=0}^{k} \binom{k}{r} (-j - 1)^{k-r}u^r \frac{t^n}{n!},$$

so

$$\begin{aligned} &t^j(u)_j (1 + t)^{u-j-1} \\ &= t^j(u)_j + \sum_{n=1}^{\infty} \sum_{k=1}^{n} s(n, k) \sum_{r=0}^{k} \binom{k}{r} (-j - 1)^{k-r} \left( \sum_{m=1}^{j} s(j, m)u^{r+m} \right) \frac{t^{n+j}}{n!}. \end{aligned} \tag{4.4}$$

Comparing coefficients of $u^k t^n/n!$ in (4.3) and (4.4), we have

$$\sum_{r=j}^{n} \frac{(-1)^{n-r}s(r, k)n!}{(r - j)!} = (n)_j \sum_{m=1}^{j} s(j, m) \sum_{i=k-m}^{n-j} \binom{i}{k - m}(-j - 1)^{i-k+m}s(n - j, i). \tag{4.5}$$

We now obtain the right-hand side of (4.5) in another way. We know

$$\begin{aligned}
&(n)_j\,(x)_{n+1}\,/(x-j)\\
&= (n)_j\,(x)_j\,(x-j-1)_{n-j}\\
&= (n)_j \sum_{m=0}^{j} s(j,\,m)x^m \sum_{i=0}^{n-j} s(n-j,\,i)(x-j-1)^i\\
&= (n)_j \sum_{m=1}^{j} s(j,\,m) \sum_{i=0}^{n-j} s(n-j,\,i)\left(\sum_{w=0}^{i}\binom{i}{w}(-j-1)^{i-w}\right)x^{w+m}. \qquad (4.6)
\end{aligned}$$

The coefficient of $x^k$ on the right side of (4.6) is

$$(n)_j \sum_{m=1}^{j} s(j,\,m) \sum_{i=k-m}^{n-j} \binom{i}{k-m}(-j-1)^{i-k+m} s(n-j,\,i),$$

which can be compared to the right side of (4.5). The left side of (4.6) can be written

$$(n)_j\,(x)_{n+1}/(x-j) = \frac{-(n)_j}{j}\,\frac{1}{1-\left(\frac{x}{j}\right)}\sum_{m=0}^{\infty} s(n+1,\,m)x^m,$$

so the coefficient of $x^k$ is

$$-(n)_j \sum_{m=1}^{k} s(n+1,\,m)\left(\frac{1}{j}\right)^{k-m+1}. \qquad (4.7)$$

Comparing (4.7) and the left side of (4.5), we have the first equality of Lemma 2.1. This completes the proof.

## REFERENCES


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