HYPERSURFACES ASSOCIATED WITH SIMSON FORMULA ANALOGUES

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1. INTRODUCTION

The *Simson formula* for the Fibonacci numbers F_n defined by

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1, \tag{1.1}$$

is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{1.2}$$

which may be expressed in determinant form as

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n. \tag{1.2$'$}$$

For the numbers w_n defined by the generalized second-order recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n,\ w_0 = a,\ w_1 = b, \tag{1.3}$$

a Simson formula was obtained in [3]. If, in this generalized Simson formula, we write $w_n = x$, $w_{n+1} = y$, then various conics—ellipses and rectangular hyperbolas—in the Euclidean plane arise as loci of the points (x, y). An analysis of these conics was made in [4] for the special cases of (1.3) which give the Fibonacci, Lucas, Pell, Fermat, and Chebyshev sequences of numbers (and also for the degenerate case when the conic breaks up).

Further developments of this theme were made by Bergum [1].

It is a natural desire to want to extend the geometrical aspect of Simson's formula (1.2) to higher dimensions. This was partly achieved in [4] for a third-order recurrence relation where a suitable analogue to Simson's formula (Waddill and Sacks [5]) was used to produce a corresponding cubic surface in three-dimensional Euclidean space. However, as this analogue had not been extended to higher-order recurrences, it was not possible to proceed to higher geometrical dimensions.

What was required was a technique, an algorithm, for determining an analogue to Simson's formula for recurrence relations of arbitrary order r.

Happily, such a method was already in existence (Hoggatt and Bicknell [2]).

After a brief, but necessary, recapitulation in the next part of this paper of the work done in [4] on the situation in three dimensions, we will proceed to employ the Hoggatt-Bicknell results [2] exclusively in the further development of our theme.

Before doing this, however, we introduce some definitions and notation.

In r-dimensional Euclidean space ($r \geqslant 2$), a locus of points whose coordinates satisfy an equation of degree m will be called a *hypersurface* of order m with dimension $r - 1$. It may be represented by the symbol L^m_{r-1}.

When the equation is linear ($m = 1$), L^1_{r-1} is the symbol for a *hyperplane* in r dimensions, i.e., a "flat" space of maximum dimension in the containing space.

2. A CUBIC SURFACE IN THREE DIMENSIONS

Consider the third-order recurrence analogue of (1.1) for the number sequence $\{P_n\}$ defined by

$$P_{n+3} = P_{n+2} + P_{n+1} + P_n \tag{2.1}$$

with initial conditions (Waddill and Sacks [5])

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1. \tag{2.2}$$

The first few numbers in this sequence are:

$$\begin{cases} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & \cdots \\ 1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 & 81 & 149 & 274 & \cdots \end{cases} \tag{2.3}$$

Waddill and Sacks [5] obtained a Simson formula analogue for $\{P_n\}$ which, not unexpectedly, was of the third degree.

Putting $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$ in their formula, the author [4] derived the cubic equation

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = 1. \tag{2.4}$$

Interpreting x, y, and z as Cartesian coordinates, we see that the points (x,y,z) lie on the cubic surface (2.4) in Euclidean space of three dimensions. For example, the point (1,1,2) in (2.3) lies on this L_2^3 (2.4), as may be easily verified.

Sections of the cubic surface (2.4) by the coordinate planes L_2^1 are the cubic curves L_1^3:

$$\begin{cases} x = 0: & 2y^3 + z^3 - 2yz^2 = 1 \\ y = 0: & x^3 + z^3 + x^2z - xz^2 = 1 \\ z = 0: & x^3 + 2y^3 + 2x^2y + 2xy^2 = 1. \end{cases} \tag{2.5}$$

A close study of these L_1^3 (2.5) might give us some insight into the nature and appearance of the L_2^3 (2.4), but no detailed investigation is undertaken here.

It must be clearly understood that the locus (2.4) and its other-dimensional analogues contain only the infinitude of points for which they are defined, i.e., within the context of this article these loci are not continuous. For instance, the point with coordinates $(0, 2^{-1/3}, 0)$ lies on the L_2^3 since $(0, 2^{-1/3}, 0)$ satisfies equation (2.4), yet the triplet $0, 2^{-1/3}, 0$ does not belong to the infinite set of numbers of the sequence $\{P_n\}$. Despite the lacunary nature of our geometrical loci, it is nevertheless sometimes worthwhile considering them as continuous entities [as for the sectional loci (2.5), for example].

In addition to the sequence (2.3) and the corresponding Simson formula analogue, Waddill and Sacks [5] discussed a closely related sequence for which the author [4] obtained a cubic equation almost identical to (2.4). However, this sequence is irrelevant to our purposes here and no further reference will be made to it. The true Fibonacci-type pattern which generalizes (1.1) and (1.2)′ is that given in (2.3), as we shall see.

Equation (2.4) of the cubic surface in three dimensions L_2^3 may also be established by a different approach using the "interesting determinant identity" of Hoggatt and Bicknell [2]. This identity, which has the structural appearance of an extension of (1.2)$'$, and which relates to the sequence (2.3) with $P_{-1} = 0$ is, in our notation,

$$\begin{vmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{vmatrix} = -1. \tag{2.6}$$

Let us now write $P_n = x$, $P_{n+1} = y$, $P_{n+2} = z$, and observe from (2.1) that

$$\begin{cases} P_{n-1} = P_{n+2} - P_{n+1} - P_n = z - x - y \\ P_{n-2} = 2P_{n+1} - P_{n+2} = 2y - z \end{cases} \tag{2.7}$$

Expanding (2.6) with the aid of (2.7), we derive

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 - 2yz^2 + x^2z - xz^2 - 2xyz = 1, \tag{2.8}$$

which is identical to equation (2.4)

Thus, the same cubic surface L_2^3 in three-dimensional Euclidean space is produced both from the Waddill and Sacks [5] cubic equation and from the Hoggatt and Bicknell [2] determinant identity.

3. HYPERSPACES IN FOUR DIMENSIONS

Next, introduce a fourth-order recurrence relation for numbers Q_n (in our notation):

$$Q_{n+4} = Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n \tag{3.1}$$

with initial conditions

$$Q_0 = 0,\ Q_1 = 1,\ Q_2 = 1,\ Q_3 = 2 \quad (Q_{-1} = 0,\ Q_{-2} = 0). \tag{3.2}$$

Then the sequence $\{Q_n\}$ looks like this:

$$\begin{cases} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & Q_8 & Q_9 & Q_{10} & \ldots \\ 1 & 1 & 2 & 4 & 8 & 15 & 29 & 56 & 108 & 208 & \ldots \end{cases} \tag{3.3}$$

Following the method by which (2.6) was established, Hoggatt and Bicknell [2] exhibited the neat determinantal identity

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = (-1)^{n+1}. \tag{3.4}$$

Write $Q_n = x$, $Q_{n+1} = y$, $Q_{n+2} = z$, $Q_{n+3} = t$. Observe that, from (3.1), we may deduce that

$$\begin{cases} Q_{n-1} = Q_{n+3} - Q_{n+2} - Q_{n+1} - Q_n = t - x - y - z \\ Q_{n-2} = 2Q_{n+2} - Q_{n+3} = 2z - t \\ Q_{n-3} = 2Q_{n+1} - Q_{n+2} = 2y - z. \end{cases} \tag{3.5}$$

Expand (3.4) along the first row. Then, the locus of the point (x,y,z,t) in four-dimensional Euclidean space is the quartic hypersurface L_3^4 (in fact, two such loci depending on the evenness or oddness of n):

$$\begin{cases} x[x\{y(t-x-y-z)-x^2\}-y\{y(2z-t)-x(t-x-y-z)\} \\ \qquad\qquad\qquad + z\{x(2z-t)-(t-x-y-z)^2\}] \\ -y[(t-x-y-z)\{y(t-x-y-z)-x^2\}-y\{y(2y-z)-x(2z-t)\} \\ \qquad\qquad\qquad + z\{x(2y-z)-(2z-t)(t-x-y-z)\}] \\ +z[(t-x-y-z)\{y(2z-y)-x(t-x-y-z)\}-x\{y(2y-z)-x(2z-t)\} \\ \qquad\qquad\qquad + z\{(2y-z)(t-x-y-z)-(2z-t)^2\}] \\ -t[(t-x-y-z)\{x(2z-t)-(t-x-y-z)^2\} \\ \qquad\qquad\qquad - x\{x(2y-z)-(2z-t)(t-x-y-z)\} \\ \qquad\qquad\qquad + y\{(2y-z)(t-x-y-z)-(2z-t)^2\}] \\ = (-1)^n. \end{cases} \tag{3.6}$$

Discretion seems the better part of valor here, so we will leave the equations in this form which is useful for deducing the sectional loci in (3.7). However, the interested reader may care to expand the expressions in (3.6) still further. It certainly bears out the author's trepidation [4] about the cumbersome algebraic manipulation involved in the fourth-order recurrence case.

Before expanding along the first row, one might secure a slightly simpler form of the determinant by adding to the fourth row the sum of the first three rows. But, in all probability, perhaps no great economy of effort in exhibiting (3.6) is thereby effected.

Planar sections (quartic curves L_1^4) of the hypersurface (3.6) by pairs of three-dimenaional coordinate hyperplanes (L_3^1) are readily obtainable, namely:

$$\begin{cases} x=0,\ y=0: & -3y^4+2z^3t+2z^2t^2-3zt^3+t^4=(-1)^n \\ x=0,\ z=0: & y^4+3y^3t-2yt^3+t^4=(-1)^n \\ x=0,\ t=0: & y^4-3y^3z-7y^2z^2-5yz^3-3z^4=(-1)^n \\ y=0,\ z=0: & -x^4-x^3t+3x^2t^2-3xt^3+t^4=(-1)^n \\ y=0,\ t=0: & -x^4-2x^3z-xz^3-3z^4=(-1)^n \\ z=0,\ t=0: & -x^4-3x^3y-4x^2y^2-2xy^3+y^4=(-1)^n. \end{cases} \tag{3.7}$$

Superficially, there does not appear to be anything memorable about these quartic plane curves.

One must be struck, in comparing (1.2)′, (2.6), and (3.4), which relate to $r = 2$, 3, and 4, respectively, by the fact that when r is even the value (± 1) of the determinant depends on the evenness or oddness of n, whereas in the case of r odd (= 3) this is not so, the value being -1 always.

These variations raise obvious questions. Is the incipient result for $r = 2, 4$ a true pattern for r even generally? Might we reasonably expect the determinantal value for $r = 5$ to be +1, and will the incipient pattern for r odd prove to be valid for r odd generally?

Answering these questions constitutes an interesting part of the overall problem.

4. HYPERSURFACES IN HIGHER DIMENSIONS

Extending the pattern of the ideas used for lower-order recurrence relations, Hoggatt and Bicknell [2] defined the sequence $\{R_n\}$ of order r by

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n \tag{4.1}$$

with initial conditions

$$R_0 = 0,\ R_1 = 1 \tag{4.2}$$

and

$$R_{-(r-2)} = R_{-(r-3)} = \cdots = R_{-1} = 0. \tag{4.3}$$

For these numbers R_n generated by the r-order recurrence relation (4.1), they established the determinantal identity

$$\begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & \cdots & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & \cdots & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)n+[(r-1)/2]}, \tag{4.4}$$

which specializes to the determinantal results (1.2)$'$, (2.6), and (3.4) already given for small values of r, namely, $r = 2$, 3, and 4, respectively. In (4,4), the notation $[(r - 1)/2]$ refers to the greatest integer function.

[It should be noted that a small typographical aberration occurs in the power of (-1) on the right-hand side of (4.4) as given in [2].]

Putting $R_n = x_1$, $R_{n+1} = x_2$, $R_{n+2} = x_3$, $\ldots$, $R_{n+r-1} = x_r$ in (4.4), and substituting by means of (4.1)-(4.3) for elements below the reverse diagonal, we could theoretically obtain the locus of points $(x_1, x_2, x_3, \ldots, x_r)$ in r-dimensional Euclidean space satisfying equation (4.4).

By analogy with (2.8) and (3.6), this locus is a L_{r-1}^r, a hypersurface (dimension $r - 1$) of order r. Sections by sets of $r - 2$ coordinate hyperplanes ("flat" hyperspaces L_{r-1}^1 of dimension $r - 1$) from the total set

$$\{x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \ldots,\ x_r = 0\}$$

of such hyperplanes give the planar curves L_1^r of order r in two dimensions corresponding to the conics (L_1^2), cubics (L_1^3), and quartics (L_1^4) in the lower-dimensional cases.

For example, in six-dimensional Euclidean space ($r = 6$), the section of the sextic hypersurface L_5^6 by the four coordinate hyperplanes $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 0$ is a plane sextic curve L_1^6.

A representative instance of (4.4) is, for $r = 5$, $n = 7$ (say),

$$\begin{vmatrix} 464 & 236 & 120 & 61 & 31 \\ 236 & 120 & 61 & 31 & 16 \\ 120 & 61 & 31 & 16 & 8 \\ 61 & 31 & 16 & 8 & 4 \\ 31 & 16 & 8 & 4 & 2 \end{vmatrix} = +1 \qquad \text{(on calculation)}$$

$$= (-1)^{28+2} = (-1)^{30} \text{ in accord with (4.4).}$$

For various values of r and n, the determinantal values in (4.4), i.e., +1 or -1, may be summarized in the following table:

Table 1: Determinantal Values in (4.4)

<table>
<tr><th rowspan="2">r / n</th><th colspan="2">Even</th><th colspan="2">Odd</th></tr>
<tr><th>2,6,10,...</th><th>4,8,12,...</th><th>3,7,11,...</th><th>5,9,13,...</th></tr>
<tr><td>Odd
1,3,5,7,...</td><td>-1</td><td>+1</td><td rowspan="2">-1</td><td rowspan="2">+1</td></tr>
<tr><td>Even
2,4,6,...</td><td>+1</td><td>-1</td></tr>
</table>

Or, expressed symbolically: If

$$r = 4k + 2,\ 4k + 3,\ 4k + 4,\ 4k + 5 \quad (k \geqslant 0),$$

then

$$(-1)^{(r-1)n+[(r-1)/2]} = (-1)^n,\ -1,\ (-1)^{n+1},\ 1,$$

respectively.

Thus, for each odd value of r, there is just one hypersurface irrespective or the value of n, while, for each even value of r, there are two "companion" hypersurfaces which depend on the evenness or oddness of n.

Now, in [4] it was stated that, when $r = 2$, a hyperbola for which n is odd (even) may be transformed into its companion hyperbola occurring when n is even (odd) by a reflection in the line $y = x$ followed by a reflection in the y-axis (x-axis).

Remembering that in two dimensions ($r = 2$), a line (a L_1^1) is a hyperplane, one may speculate whether a similar, though more complicated, system of geometrical reflections in higher even-dimensional spaces ($r = 4,6,\ldots$) will produce a transformation of one hypersurface into another. Further, one wonders whether any self-transformation of a hypersurface is possible for an odd value of r.

With these reflections, we leave the geometry.

A concluding comment on nomenclature is appropriate. Numbers, and their polynomial extensions, defined in (2.1)-(2.2), (3.1)-(3.2), and (4.1)-(4.3) are sometimes referred to in the literature as Tribonacci, Quadranacci, and r-bonacci respectively. While these adjectives are suggestive and useful, they do not appeal to the author and consequently have not been utilized in this article.

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