ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-403 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Given p, q real with $p \neq -1 - 2qk$, $k = 0, 1, 2, \ldots$, find a closed form expression for the continued fraction

$$\theta(p, q) \equiv p + \cfrac{p + q}{p + 2q + \cfrac{p + 3q}{p + 4q + \cdots}}. \tag{1}$$

HINT: Consider the *Confluent Hypergeometric* (or *Kummer*) function defined as follows:

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!}, \; b \neq 0, -1, -2, \ldots . \tag{2}$$

NOTE: $\theta(1, 1) = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cdots}}$, which was Problem H-394.

H-404 *Proposed by Andreas N. Philippou & Frosso S. Makri, Patras, Greece*

Show that

$$\text{(a)} \sum_{r=0}^{n} \sum_{i=0}^{1} \sum_{\substack{n_1, n_2 \ni \\ n_1 + 2n_2 = n - i \\ n_1 + n_2 = n - r}} \binom{n_1 + n_2}{n_1, n_2} = F_{n+2}, \; n \geq 0;$$

$$\text{(b)} \sum_{r=0}^{n} \sum_{i=0}^{k-1} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - i \\ n_1 + \cdots + n_k = n - r}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = F_{n+2}^{(k)}, \; n \geq 0, \; k \geq 2,$$

where $n_1, \ldots, n_k$ are nonnegative integers and $\{F_n^{(k)}\}$ is the sequence of Fibonacci-type polynomials of order k [1].

[1] A. N. Philippou, C. Georghiou, & G. N. Philippou, "Fibonacci-Type Polynomials of Order K with Probability Applications," *The Fibonacci Quarterly* **23**, no. 2 (1985):100-105.

H-405 *Proposed by Piero Filipponi, Rome, Italy*

(i) Generalize Problem B-564 by finding a closed form expression for

$$\sum_{n=1}^{N} [\alpha^k F_n], \quad (N = 1, 2, \ldots; \; k = 1, 2, \ldots)$$

where $\alpha = (1 + \sqrt{5})/2$, F_n is the n^{th} Fibonacci number, and $[x]$ denotes the greatest integer not exceeding x.

(ii) Generalize the above sum to negative values of k.

(iii) Can this sum be further generalized to any rational value of the exponent of α?

Remark: As to (iii), it can be proved that

$$[\alpha^{1/k} F_n] = F_n, \text{ if } 1 \leqslant n \leqslant [(\ln \sqrt{5} - \ln(\alpha^{1/k} - 1))/\ln \alpha].$$

References

1. V. E. Hoggatt, Jr., & M. Bicknell-Johnson, "Representstion of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio," *The Fibonacci Quarterly* **17**, no. 4 (1979):306-318.
2. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Boston: Houghton Mifflin Company, 1969).

SOLUTIONS

Sum Zeta!

H-381 *Proposed by Dejan M. Petkovic, Nis, Yugoslavia*
(Vol. 23, no. 1, February 1985)

Let N be the set of all natural numbers and let $m \in N$. Show that

(i) $$\zeta(2m - 2) = \frac{(-)^m \bar{u}^{2m-2}(m - 1)}{(2m - 1)!} + \sum_{i=2}^{m-1} \frac{(-)^i \bar{u}^{2i-2}}{(2i - 1)!} \cdot \zeta(2m - 2i), \; m \geqslant 2,$$

(ii) $$\beta(2m - 1) = \sum_{i=1}^{m-1} \frac{(-)^i \bar{u}^{2i}}{2^{2i}(2i)!} \cdot \beta(2m - 2i - 1), \; m \geqslant 2,$$

(iii) $$\zeta(2m) = \frac{2^{2m}}{2^{2m} - 1} \sum_{i=0}^{m-1} \frac{(-)^i \bar{u}^{2i+1}}{2^{2i+1}(2i + 1)!} \cdot \beta(2m - 2i - 1), \; m \geqslant 1,$$

where

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m}, \; m \geqslant 2, \text{ are Riemann zeta numbers}$$

and

$$\beta(m) = \sum_{n=1}^{\infty} (-)^{n-1}(2 \quad - 1)^{-m}, \; m \geqslant 1.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

We use the known expressions

$$\zeta(2m) = \frac{(2\overline{u})^{2m}}{2(2m)!}(-1)^{m-1}B_{2m}, \quad m = 1, 2, \ldots, \tag{1}$$

$$\beta(2m+1) = \frac{(\overline{u}/2)^{2m+1}}{2(2m)!}(-1)^m E_{2m}, \; m = 0, 1, 2, \ldots,$$

where $\overline{u}$ denotes the constant π, and the B_{2m} and E_{2m} are the Bernoulli and Euler numbers, respectively.

These may be defined by the following generating functions:

$$x \cot x = \sum_{m=0}^{\infty} B_{2m}(-1)^m \frac{(2x)^{2m}}{(2m)!}, \tag{3}$$

and

$$\sec x = \sum_{m=0}^{\infty} E_{2m}(-1)^m \frac{x^{2m}}{(2m)!}. \tag{4}$$

Setting $x = \overline{u}z$, then

$$\overline{u}z \cot \overline{u}z = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!}(2\overline{u}z)^{2m} \cdot \frac{2(2m)!\zeta(2m)(-1)^{m-1}}{(2\overline{u})^{2m}},$$

or

$$\overline{u}z \cot \overline{u}z = 1 - 2\sum_{m=1}^{\infty} \zeta(2m)z^{2m}. \tag{5}$$

Also,

$$\sec \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m}}{(2m)!} \cdot \frac{2(2m)!(-1)^m}{(\overline{u}/2)^{2m+1}} \beta(2m+1),$$

or

$$\sec \overline{u}z = \frac{4}{\overline{u}} \sum_{m=0}^{\infty} \beta(2m+1)(2z)^{2m}. \tag{6}$$

We also use the following well-known expressions:

$$\sin \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!}; \tag{7}$$

$$\cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m}}{(2m)!}. \tag{8}$$

Multiplying (5) and (7), we obtain:

$$\overline{u}z \cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!} - 2\sum_{m=1}^{\infty} z^{2m+1} \sum_{i=0}^{m-1} (-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!} \zeta(2m-2i);$$

on the other hand, from (8),

$$\overline{u}z \cos \overline{u}z = \sum_{m=0}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m)!}.$$

Thus,

$$\sum_{m=1}^{\infty} (-1)^m \frac{(\overline{u}z)^{2m+1}}{(2m+1)!}(1 - (2m+1)) = 2\sum_{m=1}^{\infty} z^{2m+1} \sum_{i=0}^{m-1} (-1)^m \frac{(\overline{u})^{2i+1}}{(2i+1)!} \zeta(2m-2i).$$

Comparing coefficients,

$$\frac{-2m(-1)^m(\overline{u})^{2m+1}}{(2m+1)!} = 2\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!}\zeta(2m-2i):$$

replacing m by $m-1$ and dividing by $2\overline{u}$ yields:

$$\begin{aligned}\frac{(m-1)(-1)^m(\overline{u})^{2m-2}}{(2m-1)!} &= \sum_{i=0}^{m-2}(-1)^i \frac{(\overline{u})^{2i}}{(2i+1)!}\zeta(2m-2i-2)\\ &= -\sum_{i=1}^{m-1}(-1)^i \frac{(\overline{u})^{2i-2}}{(2i-1)!}\zeta(2m-2i)\\ &= \zeta(2m-2) - \sum_{i=2}^{m-1}(-1)^i \frac{(\overline{u})^{2i-2}}{(2i-1)!}\zeta(2m-2i).\end{aligned}$$

This is equivalent to the result indicated in (i).

Multiplying (6) and (8), we obtain:

$$1 = 4/\overline{u}\sum_{m=0}^{\infty} z^{2m}\sum_{i=0}^{m}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!} 2^{2m-2i}\beta(2m-2i+1);$$

hence, for $m \geqslant 1$,

$$0 = \sum_{i=0}^{m}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!} 2^{2m-2i}\beta(2m-2i+1).$$

Replacing m by $m-1$ and dividing by 2^{2m-2} yields:

$$\begin{aligned}0 &= \sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!} 2^{-2i}\beta(2m-2i-1)\\ &= \beta(2m-1) + \sum_{i=1}^{m-1}(-1)^i \frac{(\overline{u})^{2i}}{(2i)!} 2^{-2i}\beta(2m-2i-1).\end{aligned}$$

This last result corrects (ii), which is incorrect in the sign of one of its members.

Finally, multiplying (6) and (7) yields:

$$\tan \overline{u}z = 4/\overline{u}\sum_{m=1}^{\infty} z^{2m-1}\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!} 2^{2m-2i-2}\beta(2m-2i-1).$$

On the other hand, since $\tan x = \cot x - 2\cot 2x$, we have:

$$\begin{aligned}\tan \overline{u}z &= (\overline{u}z)^{-1}(\overline{u}z \cot \overline{u}z - 2\overline{u}z \cot 2\overline{u}z)\\ &= (\overline{u}z)^{-1}\left\{1 - 2\sum_{m=1}^{\infty}\zeta(2m)z^{2m} - 1 + 2\sum_{m=1}^{\infty}\zeta(2m)(2z)^{2m}\right\}\\ &= 2/\overline{u}\sum_{m=1}^{\infty}\zeta(2m)(2^{2m}-1)z^{2m-1}.\end{aligned}$$

Comparing coefficients,

$$4/\overline{u}\sum_{i=0}^{m-1}(-1)^i \frac{(\overline{u})^{2i+1}}{(2i+1)!} 2^{2m-2i-2}\beta(2m-2i-1) = 2/\overline{u}\zeta(2m)(2^{2m}-1),$$

or, equivalently:

$$\zeta(2m) = \frac{2^{2m}}{2^{2m-1}} \sum_{i=0}^{m-1} (-1)^i (\overline{u}/2)^{2i+1} \frac{\beta(2m-2i-1)}{(2i+1)!},$$

which is (iii). Q.E.D.

Also solved by C. Georghiou, S. Papastavridis, P. Siafarikas, P. Sypsas, and the proposer.

H-382 *Proposed by Andreas N. Philippou, Patras, Greece (Vol. 23, no. 1, February 1985)*

For each fixed positive integer k, define the sequence of polynomials $A_{n+1}^{(k)}(p)$ by

$$A_{n+1}^{(k)}(p) = \sum_{n_1+\cdots+n_k} \binom{n_1+\cdots+n}{n_1, \ldots, n} \left(\frac{1-p}{p}\right)^{n_1, \ldots, n_k} \quad (n \geqslant 0, -\infty < p < \infty), \tag{1}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n + 1$. Show that

$$A_{n+1}^{(k)}(p) \leqslant (1-p)p^{-(n+1)}(1-p^k)^{[n/k]} \quad (n \geqslant k-1,\ 0 < p < 1), \tag{2}$$

where $[n/k]$ denotes the greatest integer in (n/k).

It may be noted that (2) reduces to

$$F_n \leqslant 2^n \left(\frac{2^k-1}{2^k}\right)^{[n/k]} \quad (n \geqslant k-1) \tag{3}$$

and

$$F_n \leqslant 2^n (3/4)^{[n/2]} \quad (n \geqslant 1), \tag{4}$$

where $\{F_n^{(k)}\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ denote the Fibonacci sequence of order k and the usual Fibonacci sequence, respectively, if $p = 1/2$ and $p = 1/2$, $k = 2$.

References

1. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly 2*, no. 2 (1964):154.
2. A. N. Philippou. Problem H-322. *The Fibonacci Quarterly 19*, no. 1 (1981): 93.

Solution by the proposer

For each fixed positive integer k, let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ $(x > 0)$ be the sequence of Fibonacci-type polynomials of order k [4] and denote by L_n and $N_n^{(k)}$ the longest success run and the number of success runs of order k, respectively, in n Bernoulli trials. It follows from the definition of $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ that

$$F_2^{(k)}(x) = x$$

and

$$F_{n+2}^{(k)}(x) = (1+x)F_{n+1}^{(k)}(x) = \cdots = (1+x)^n F_2^{(k)}(x) = (1+x)^n x \quad (1 \leqslant n \leqslant k-1,\ x > 0),$$

which gives

$$F_{n+2}^{(k)}((1-p)/p) = (1-p)p^{-(n+1)} \quad (0 \leqslant n \leqslant k-1,\ 0 < p < 1). \tag{5}$$

Furthermore,

$$\begin{aligned} F_{n+2}^{(k)}((1-p)/p) &= (1-p)p^{-(n+1)}P(L_n \leqslant k-1) \\ &\qquad\qquad (n \geqslant k-1,\ 0 < p < 1), \qquad (6) \\ &\qquad\qquad \text{by Theorem 2.1(a) of [4],} \\ &= (1-p)p^{-(n+1)}P(N_n^{(k)} = 0), \\ &\qquad\qquad \text{by the definition of } L_n \text{ and } N_n^{(k)}, \\ &= (1-p)p^{-(n+1)}\{1 - P(N_n^{(k)} \geqslant 1)\} \\ &\leqslant (1-p)p^{-(n+1)}\{1 - \{1 - (1-p^k)^{[n/k]}\}\}, \\ &\qquad\qquad \text{by Proposition 6.3 of [1],} \\ &= (1-p)p^{-(n+1)}(1-p^k)^{[n/k]}. \end{aligned}$$

But

$$F_{n+2}^{(k)}((1-p)/p) = A_{n+1}^{(k)}(p) \quad (n \geqslant 0,\ 0 < p < 1), \tag{7}$$

by Lemma 2.2(b) of [4] and (1).

Relations (5)-(7) establish (2), which reduces to (3) and (4), respectively, since

$$F_{n+2}^{(k)} = F_{n+2}^{(k)}(1) = A_{n+1}^{(k)}(1/2) \quad (n \geqslant 0)$$

and

$$F_{n+2}^{(2)} = F_n \quad (n \geqslant 0).$$

It may be noted that inequalities (3) and (4) are sharper than those given in [2] and [3], respectively.

References

1. S. M. Berman. *The Elements of Probability*. Reading, Mass.: Addison-Welesly, 1969.
2. J. A. Fuchs. Problem B-39. *The Fibonacci Quarterly* **2**, no. 2 (1964):154.
3. A. N. Philippou. Problem H-322. *The Fbionacci Quarterly* **19**, no. 1 (1981): 93.
4. A. N. Philippou and F. S. Makri. "Longest Success Runs and Fibonacci-Type Polynomials." *The Fibonacci Quarterly* **23**, no. 4 (1985):338-346.

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