ON THE EXISTENCE OF e-MULTIPERFECT NUMBERS

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Dedicated to the memory of Robert Arnold Smith

1. INTRODUCTION

By an exponential divisor (or e-divisor) of a positive integer $N > 1$ with canonical form

$$N = p_1^{a_1} \ldots p_r^{a_r},$$

we mean a divisor d of N of the form

$$d = p_1^{b_1} \ldots p_r^{b_r},\ b_i | a_i,\ i = 1, \ldots, r.$$

The sum of such divisors of N is denoted by $\sigma^{(e)}(N)$, and the number of such divisors by $\tau^{(e)}(N)$. By convention, 1 is an exponential divisor of itself, so that $\sigma^{(e)}(1) = 1$. The functions $\tau^{(e)}(N)$ and $\sigma^{(e)}(N)$ were introduced in [1] and have been studied in [1] and [2].

An integer N is said to be e-perfect whenever $\sigma^{(e)}(N) = 2N$, and e-multiperfect when $\sigma^{(e)}(N) = kN$ for an integer $k > 2$. In [1] and [2], several examples of e-perfect numbers are given. It is also proved in [2] that all e-perfect and all e-multiperfect numbers are even.

Several unsolved problems are listed in [2], and one of them is whether or not there exists an e-multiperfect number. In this paper, we show that if such a number exists, it must indeed be very, very large.

2. NOTATION AND SOME LEMMAS

In all that follows, the positive integer N is assumed to be an e-multiperfect number, so that

$$\sigma^{(e)}(N) = kN \text{ for some integer } k > 2. \tag{2.1}$$

Note that if n is a square-free integer, then $\sigma^{(e)}(n) = n$, so that if $(n, N) = 1$, then Nn is also e-multiperfect. Hence, we assume (as we may) in the future that N is powerful. Also note here that we have used the fact that $\sigma^{(e)}$ is a multiplicative function.

Write

$$N = 2^h(q_1^{a_1} \ldots q_s^{a_s})(p_1^{b_1} \ldots p_t^{b_t}), \tag{2.2}$$

where the p's and q's are distinct primes, and each a_i is a non-square integer $\geqslant 2$, and each b_j is a square integer $\geqslant 4$. It follows then that each $\sigma^{(e)}(q_i^{a_i})$ is even and each $\sigma^{(e)}(p_j^{b_j})$ is odd.

Let $k = 2^{\omega}M$, where M is odd and $\omega \geqslant 0$.

Lemma 2.3: N is even, i.e., $h \geqslant 2$.

This is a consequence of Theorem 2.2 of [2].

Lemma 2.4: $s < \omega + h$.

Proof: The relation $\sigma^{(e)}(N) = kN$ gives

$$\sigma^{(e)}(2^h)\left[\prod_{i=1}^{s}\sigma^{(e)}(q_i^{a_i})\right]\left[\prod_{j=1}^{t}\sigma^{(e)}(p_j^{b_j})\right] = 2^{\omega+h}M(q_1^{a_1}\cdots q_s^{a_s})(p_1^{b_1}\cdots p_t^{b_t}).$$

Since the only even factors on the left side are $\sigma^{(e)}(q_1^{a_1}), \ldots, \sigma^{(e)}(q_s^{a_s})$, and since $2|\sigma^{(e)}(2^h)$, the result follows.

In what follows, the letter p represents a prime.

Lemma 2.5:

$$\prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) < (1.27885)\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right).$$

Remark: This is a stronger form of Lemma 2.1 of [2], where a similar result is proved with the multiplicative constant on the right being $27/16 \approx 1.6875$. For our present purpose, we need the above stronger result.

Proof of Lemma 2.5:

$$\prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) < \prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2}\right)\left(1 + \frac{1}{p^3}\right)$$

$$= \prod_{p \neq 2,\, q_1, \ldots, q_s}\left(1 + \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^4}\right)\left(1 + \frac{1}{p^3}\right)$$

$$< \left[\zeta(2)\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right)\right]$$

$$\cdot \left[\zeta(4)\left(1 - \frac{1}{2^4}\right)\left(1 - \frac{1}{q_1^4}\right)\cdots\left(1 - \frac{1}{q_s^4}\right)\right]^{-1}$$

$$\cdot \left[\zeta(3)\left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{q_1^3}\right)\cdots\left(1 - \frac{1}{q_s^3}\right)\right]$$

$$< \frac{7}{10}\ \frac{\zeta(2)\zeta(3)}{\zeta(4)}\left(1 - \frac{1}{q_1^2}\right)\cdots\left(1 - \frac{1}{q_s^2}\right),$$

on utilizing the result that

$$\left[1 - \frac{1}{q_j^3}\right]\left[1 - \frac{1}{q_j^4}\right]^{-1} < 1,\ j = 1, \ldots, s.$$

Using

$$\zeta(2) < 1.64494,\ \zeta(3) < 1.20206, \text{ and } \zeta(4) < 1.08232$$

([3], p. 811), we obtain the proof of the lemma.

Lemma 2.6:

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(h-2)/2}}\right)\left[\left(1 + \frac{1}{q_1}\right)\left(1 - \frac{1}{q_1^2}\right) \cdots \left(1 + \frac{1}{q_s}\right)\left(1 - \frac{1}{q_s^2}\right)\right],$$

where $1 + 2^{(h-2)/2}$ is to be taken as $1 + \frac{1}{2}$ for $h = 2, 3$.

Proof:

$$k = \frac{\sigma^{(e)}(N)}{N} = \frac{\sigma^{(e)}(2^h)}{2^h} \cdot \left[\prod_{i=1}^{r} \frac{\sigma^{(e)}(p_i^{a_i})}{p_i^{a_i}}\right]\left[\prod_{j=1}^{s} \frac{\sigma^{(e)}(q_j^{b_j})}{q_j^{b_j}}\right].$$

We note first that, for any prime p, we have

$$\frac{\sigma^{(e)}(p^m)}{p^m} \quad \frac{\sigma^{(e)}(p^2)}{p^2} = 1 + \frac{1}{p}, \; m = 2, 3, \ldots .$$

Also, for $m \geqslant 2$,

$$\begin{aligned}\frac{\sigma^{(e)}(p^m)}{p^m} &\leqslant (p^m + p^{m/2} + p^{m/3} + \cdots + p)/p^m \\ &< 1 + \frac{1}{p^{m/2}} + \frac{1}{p^{m/2+1}} + \frac{1}{p^{m/2+2}} + \cdots \\ &= 1 + \frac{1}{p^{(m/2-1)}(p-1)}.\end{aligned}$$

Thus,

$$\frac{\sigma^{(e)}(2^h)}{2^h} < 1 + \frac{1}{2^{(h/2)-1}} \text{ for } h \geqslant 4; \; \frac{\sigma^{(e)}(2^h)}{2^h} \leqslant 1 + \frac{1}{2}, \; h = 2, 3,$$

and

$$\frac{\sigma^{(e)}(q_j^{b_j})}{q_j^{b_j}} \leqslant 1 + \frac{1}{q_j}, \; j = 1, 2, \ldots, s.$$

Next,

$$\begin{aligned}\prod_{i=1}^{r}\left[\frac{\sigma^{(e)}(p_i^{a_i})}{p_i^{b_i}}\right] &\leqslant \prod_{i=1}^{r} \frac{\sigma^{(e)}(p_i^4)}{p_i^4} = \prod_{i=1}^{r}\left(1 + \frac{1}{p_i^2} + \frac{1}{p_i^3}\right) \\ &\leqslant \prod_{p \neq 2,\, q_1,\, \ldots,\, q_s}\left(1 + \frac{1}{p^2} + \frac{1}{p^3}\right) \\ &< (1.27885)\left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q_s^2}\right),\end{aligned}$$

on using Lemma 2.5. The result (2.6) now follows.

3. MAIN RESULTS

Given $k \geqslant 3$, we shall estimate h and s as functions of k and show that

$$\lim_{k\to\infty} h = \lim_{k\to\infty} s = \infty. \tag{3.1}$$

These follow from the results $\omega \leqslant \log k/\log 2$ and

$$h \geqslant s - \omega \geqslant [(1 - \log(32/27)/\log 2)\log k - \log((1.27885)(1.5))]/\log(32.27). \tag{3.2}$$

To obtain (3.2), we utilize Lemmas 2.4 and 2.6. Thus,

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2}\right)\prod_{i=1}^{s}\left(1 + \frac{1}{q_i}\right)\left(1 - \frac{1}{q_i^2}\right). \tag{3.3}$$

If we take logarithms of both sides and use the estimate that, for all i,

$$\left(1 + \frac{1}{q_i}\right)\left(1 - \frac{1}{q_i^2}\right) \leqslant \left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{3^2}\right) = \frac{32}{27}, \tag{3.4}$$

then, after carrying out routine calculations, we get (3.2) from (3.3).

Actually, the estimate for h in (3.2) can be vastly improved as shown below.

Let $H_0 = H_0(k)$ be the smallest value of h for which N, given by (2.2), is a solution of (2.1). Then we shall show that H_0 increases exponentially with k. In fact, there is a function $H(k)$ such that $H_0(k) \geqslant H(k)$ and $\log\log H \sim \log k$ as $k \to \infty$.

Let $Q_1 = 3$, $Q_2 = 5$, ... be the sequence of odd primes. From (3.2), we have

$$\frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(H-2)/2}}\right)\prod_{i=1}^{H+\omega}\left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right) \tag{3.5}$$

Now let H be the smallest integer satisfying (3.5), so

$$\begin{aligned}\left(1 + \frac{1}{2^{(H-3)/2}}\right)\prod_{i=1}^{H-1+\omega}\left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right) &< \frac{k}{1.27885} \\ \leqslant \left(1 + \frac{1}{2^{(H-2)/2}}\right)\prod_{i=1}^{H+\omega}\left(1 + \frac{1}{Q_i}\right)&\left(1 - \frac{1}{Q_i^2}\right).\end{aligned} \tag{3.6}$$

It is clear that $H_0(k) \geqslant H(k)$.

Theorem 3.7: $\log\log H \sim \log k \; (k \to \infty)$.

Proof: Taking logarithms and letting $k \to \infty$ and noting that

$$\log(1 + 2^{-(H-2)/2}) \leqslant \log\left(1 + \frac{1}{2}\right) = O(1) \;\; (H \to \infty),$$

and similarly for $\log(1 + 2^{-(H-3)/2})$, and using the result

$$\sum_{i=1}^{t}\log\left(1 - \frac{1}{Q_i^2}\right) = O(1), \; t \to \infty,$$

we get

$$\sum_{i=1}^{H+\omega-1} \log\left(1 + \frac{1}{Q_i}\right) + 0(1) \leqslant \log k + 0(1)$$
$$\leqslant \sum_{i=1}^{H+\omega} \log\left(1 + \frac{1}{Q_i}\right) + 0(1). \tag{3.8}$$

Note that as $k \to \infty$, $H \to \infty$, and

$$\sum_{i=1}^{H} \log\left(1 + \frac{1}{Q_i}\right) \sim \log \log H \ (H \to \infty).$$

Thus, (3.8) gives

$$\log \log(H + \omega) \sim \log k \ (k \to \infty).$$

Since $\omega = 0\ (\log k)$, this gives

$$\log \log H \sim \log k \ (k \to \infty). \tag{3.9}$$

Explicit Lower Bounds for N

We shall now give some explicit lower bounds for $N(k)$, the smallest value of N for given values of k that satisfies (2.1).

First, we note the explicit values of $H = H(k)$ for certain small values of k.

Lemma 3.10:

(i) $H(3) = 4$

(ii) $H(4) = 41$

(iii) $H(5) = 135$

(iv) $H(6) = 426$

(v) $H(7) = 1382$

(vi) $H(8) = 4553$

Proof: We recall the definition of H and utilize its characterization given by (3.6). Then a computer calculation gives the above results.

Lemma 3.11: Let $P(x)$ denote the product of all the primes not exceeding x. Then

(i) $\log P(x) > .84x$ for $x \geqslant 101$,

(ii) $\log P(x) > .98x$ for $x \geqslant 7481$.

This follows from Theorem 10 of the estimates given by Rosser and Schoenfeld [4].

Of course, the Prime Number Theorem gives the result that $\log P(x) \sim x$.

Theorem 3.12:

$$N(3) > 2 \cdot 10^{7} \tag{3.13}$$

$$N(4) > 10^{85} \tag{3.14}$$

$$N(5) > 10^{320} \tag{3.15}$$

$N(6) > 10^{1210}$; also $N(k) > 10^{1210}$ for all even k for which $\omega = \omega(k) = 1$. (3.16)

$N(k) > 10^{5270}$ for all odd $k \geqslant 7$. (3.17)

$N(k) > 10^{19884}$ for all even $k \geqslant 8$, for which $\omega = \omega(k) = 3$. (3.18)

Proof: We shall use the results of Lemmas 3.10 and 3.11. We shall illustrate the proof by considering only a few cases.

Let

$$G(H,\ u) = \left(1 + \frac{1}{2^{(H-2)/2}}\right) \prod_{i=1}^{u} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right). \tag{3.19}$$

(i) <u>$k = 3$</u>: Since $H(3) = 4$, by Lemma 2.5 and (3.6), we should have

$$G(3,\ u) \geqslant 3/1.27885.$$

A computer run shows that the smallest value of u for which this inequality holds is $u = 4$. Hence, $s \geqslant 4$ and

$$N(3) \geqslant 2^4 \prod_{i=1}^{4} Q^2 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 = 21344400 > 2 \cdot 10^7.$$

(ii) <u>$k = 7$</u>: Since $H(7) = 1382$, $(H - 2)/2 = 691$. We should then have

$$G(7,\ u) \geqslant 7/1.27885.$$

A computer run shows that the smallest u that satisfies this is $u = 1382$. Thus,

$$N(7) \geqslant 2^{1382} \prod_{i=1}^{1382} Q_i^2 > 10^{5270}$$

on using Lemma 3.11.

(iii) <u>k odd > 7</u>: Then $H(k)$ satisfies

$$k/(1.27885) < \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

Since $7/1.27885 < k/1.27885$, we have $H(k) > H(7) = 1382$. Hence, the value of u that satisfies

$$G(k,\ u) > k/1.27885$$

is > 1382, and $N(k) > 10^{5270}$ for all odd $k > 7$.

(iv) <u>$k = 8$</u>: We have $\omega = 3$ and $H = H(8) = 4553$. Thus, $(H - 2)/2 = 2276.5$ and

$$\frac{8}{1.27885} \leqslant \left(1 + \frac{1}{2^{2276.5}}\right) \prod_{i=1}^{4553} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

A computer run shows that the smallest value of u for which

$$G(8,\ u) \geqslant 8/1.27885$$

is $u = 4556$. Hence, $s \geqslant 4556$ and

$$N(8) \geqslant 2^{4553} \prod_{i=1}^{4556} Q^2 > 10^{19884}.$$

on using Lemma 3.11 and a computer calculation.

(v) <u>k even and > 8 and $\omega = \omega(k) = 3$</u>: We have

$$\frac{8}{1.27885} < \frac{k}{1.27885} \leqslant \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)+\omega} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right)$$

$$= \left(1 + \frac{1}{2^{(H(k)-2)/2}}\right) \prod_{i=1}^{H(k)+3} \left(1 + \frac{1}{Q_i}\right)\left(1 - \frac{1}{Q_i^2}\right).$$

From this, it is clear that $H(k) > H(8)$ for all even k for which $\omega = \omega(k) = 3$.

Remark 3.20: Though we are unable to prove this, it is very likely that $H(k)$ increases monotonically with k for all $k \geqslant 3$. The numerical evidence supports this; therefore, we make the following conjectures.

Conjecture 3.21: $H(k)$ and $H_0(k)$ are monotonic functions of k for $k \geqslant 3$.

Conjecture 3.22: There are no e-multiperferfect numbers.

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