AN UPPER BOUND FOR THE GENERAL RESTRICTED PARTITION PROBLEM

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The function $p^*(p_1, p_2, \ldots, p_m; n)$ is defined as the number of partitions of the integer n into at most m positive integers $p_1, p_2, \ldots, p_m$, where the order is irrelevant. An upper bound for the number of partitions is given. This upper bound is then compared with two known particular cases. An upper bound for the function $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$ is also given. This last function represents the number of partitions of all integers between 0 and n into at most m positive integers $p_1, p_2, \ldots, p_m$.

1. INTRODUCTION

The number of partitions as defined above is equal to the number of solutions of the Diophantine equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n$$

in integers $x_i \geqslant 0$, where the p_i are given positive integers which need not be distinct. If $(p_1, p_2, \ldots, p_m) = d > 1$, then $p^*(p_1, p_2, \ldots, p_m; n) = 0$ unless d divides n, in which case the factor d can be removed from the above equation without altering the number of partitions. That is,

$$p^*(p_1, p_2, \ldots, p_m; n) = p^*\left(\frac{p_1}{d}, \frac{p_2}{d}, \ldots, \frac{p_m}{d}; \frac{n}{d}\right),$$

where

$$\left(\frac{p_1}{d}, \frac{p_2}{d}, \ldots, \frac{p_m}{d}\right) = 1 \text{ when } d/n.$$

Thus, we can assume that the equation is reduced and that $(p_1, p_2, \ldots, p_m) = 1$ for the rest of this paper. We can also assume without loss of generality that

$$p_1 \leqslant p_2 \leqslant p_3 \leqslant \ldots \leqslant p_m,$$

where there must be at least one strict inequality if $(p_1, p_2, \ldots, p_m) = 1$ unless $p_1 = p_2 = \cdots = p_m = 1$. The number of partitions of n into exactly the parts $p_1, p_2, \ldots, p_m$ will be denoted by the function

$$p(p_1, p_2, \ldots, p_m; n).$$

This is equal to the number of solutions of the equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n$$

in integers $x_i \geqslant 1$.

It is known that

$$p(p_1, p_2, \ldots, p_m; n) = p^*(p_1, p_2, \ldots, p_m; n - (p_1 + p_2 + \cdots + p_m)) \tag{1.1}$$

and that the function $p^*(p_1, p_2, \ldots, p_m; n)$ satisfies the recurrence equation

$$\begin{aligned} p^*(p_1, p_2, \ldots, p_m; n) - p^*(p_1, p_2, \ldots, p_m; n - p_m) \\ = p^*(p_1, p_2, \ldots, p_{m-1}; n), \end{aligned} \tag{1.2}$$

where $p^*(p_1, p_2, \ldots, p_m; 0) = 1$.

2. PRELIMINARY RESULTS

In order to determine an upper bound for $p^*(p_1, p_2, \ldots, p_m; n)$ under the most general possible condition, which is $(p_1, p_2, \ldots, p_m) = 1$, we require some preliminary results, which will be stated without proof. The proofs are quite straightforward but in the case of (2.1) rather lengthy. The proofs have been omitted in this revised version to reduce the length of the paper.

If $(p_1, p_2) = \alpha_2$ and $(p_1, p_2, p_3) = \alpha_3$, then, for $n \geqslant 0$,

$$p^*(p_1, p_2, p_3; n) \leqslant \frac{\alpha_3}{2p_1p_2p_3}\left(n + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2 p_3}{\alpha_3}\right)\right)^2. \tag{2.1}$$

If $A > 0$ and $B > 0$ and k is an integer $\geqslant 2$, then

$$\sum_{r=0}^{t}(Ar + B)^{k-1} \leqslant \frac{1}{Ak}\left(A\left(t + \frac{1}{2}\right) + B\right)^k. \tag{2.2}$$

The upper bound in (2.1) cannot be weakened, since it is actually attained under very special circumstances. If we consider $p^*(p_1, p_2, p_3; n)$, where

$$(p_1, p_2, p_3) = 1 \quad \text{and} \quad p_3 = \frac{2p_1p_2}{\alpha_2^2},$$

then $\alpha_3 = 1$ and, for an arbitrary positive integer k, we have, using (2.1), that

$$p^*\left(p_1, p_2, \frac{2p_1p_2}{\alpha_2^2}; \frac{2kp_1p_2}{\alpha_2}\right) \leqslant (k + 1)^2.$$

But it can be shown that in this case we have

$$p^*\left(p_1, p_2, \frac{2p_1p_2}{\alpha_2^2}; \frac{2kp_1p_2}{\alpha_2}\right) = (k + 1)^2$$

and the bound is attained.

3. THE MAIN RESULT

We now state and prove the main result of this paper.

Theorem: If $(p_1, p_2) = \alpha_2$, $(p_1, p_2, p_3) = \alpha_3, \ldots, (p_1, p_2, \ldots, p_m) = \alpha_m$, then for $n \geqslant 0$ and $m \geqslant 3$,

$$p^*(p_1, p_2, \ldots, p_m; n) \tag{3.1}$$
$$\leqslant \frac{\alpha_m}{p_1 p_2 \ldots p_m (m-1)!}\left(n + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{m-1}}{\alpha_m}p_m\right)\right)^{m-1},$$

where, if the partition is reduced, $\alpha_m = 1$.

Proof: Assume the result correct if $m = k$ (say), and consider

$$p^*(p_1, p_2, \ldots, p_{k+1}; n), \text{ where } (p_1, p_2, \ldots, p_{k+1}) = 1.$$

Writing

$$n = ap_{k+1} + b, \text{ where } a = \left[\frac{n}{p_{k+1}}\right] \text{ and } 0 \leqslant b \leqslant p_{k+1} - 1.$$

$$\therefore\ p^*(p_1, p_2, \ldots, p_k, p_{k+1}; n) = \sum_{i=0}^{a} p^*(p_1, p_2, \ldots, p_k; ip_{k+1} + b),$$

using (1.2), since $p^*(p_1, p_2, \ldots, p_{k+1}; b) = p^*(p_1, p_2, \ldots, p_k; b)$, where $(p_1, p_2, \ldots, p_k) = \alpha_k$. Now the sum is zero if $\alpha_k \nmid ip_{k+1} + b$. Consider

$$ip_{k+1} + b \equiv 0 \pmod{\alpha_k}, \text{ where } (p_{k+1}, \alpha_k) = 1$$

as $(p_1, p_2, \ldots, p_k, p_{k+1}) = 1$. Thus, there is a unique solution

$$i = i_0 \pmod{\alpha_k}, \text{ where } 0 \leqslant i_0 \leqslant \alpha_k - 1.$$

$$\therefore\ i = i_0,\ i_0 + \alpha_k,\ i_0 + 2\alpha_k,\ \ldots,\ i_0 + \left[\frac{a - i_0}{\alpha_k}\right]\cdot\alpha_k \leqslant a \text{ if } a - i_0 \geqslant 0.$$

Hence,

$$p^*(p_1, p_2, \ldots, p_{k+1}; n)$$

$$= \begin{cases} \displaystyle\sum_{i \text{ (as above)}} p^*(p_1, p_2, \ldots, p_k; ip_{k+1} + b) & \text{if } a - i_0 \geqslant 0 \\ = 0 & \text{if } a - i_0 < 0 \end{cases}$$

$$\leqslant \sum_i \frac{\alpha_k}{p_1 p_2 \cdots p_k (k-1)!}\left(ip_{k+1} + b + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_{k\mp 1}}{\alpha_k}p_k\right)\right)^{k-1}$$

$$= \frac{\alpha_k}{p_1 \cdots p_k (k-1)!}\sum_{r=0,1,2,\ldots}^{\left[\frac{a-i_0}{\alpha_k}\right]}\left(p_{k+1}(i_0 + r\alpha_k) + b + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k}p_k\right)\right)^{k-1}$$

$$= \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \sum_{r=0}^{\left[\frac{a-i_0}{\alpha_k}\right]} \left(p_{k+1}\alpha_k r + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k\right)\right)^{k-1}$$

$$= \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \sum_{r=0,1,2,\ldots}^{t} (Ar + B)^{k-1}, \text{ where } t = \left[\frac{a - i_0}{\alpha_k}\right] \geqslant 0$$

and $A = p_{k+1}\alpha_k$, $B = p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k\right)$,

$$\leqslant \frac{\alpha_k}{p_1 \cdots p_k (k-1)!} \cdot \frac{1}{p_{k+1}\alpha_k \cdot k}\left(p_{k+1}\alpha_k\left(t + \frac{1}{2}\right) + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k\right)\right)^k, \text{ using 2.2,}$$

$$= \frac{1}{p_1 p_2 \cdots p_{k+1} k!}\left(p_{k+1}\alpha_k t + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k + \frac{\alpha_k}{1} p_{k+1}\right)\right)^k.$$

Now $\alpha_k p_{k+1} t = p_{k+1}\alpha_k \left[\frac{a - i_0}{\alpha_k}\right] \leqslant p_{k+1}(a - i_0)$

$$\leqslant \frac{1}{p_1 p_2 \cdots p_{k+1} k!}\left(a p_{k+1} - p_{k+1} i_0 + p_{k+1} i_0 + b + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \cdots + \frac{\alpha_k}{1} p_{k+1}\right)\right)^k$$

$$= \frac{1}{p_1 p_2 \cdots p_{k+1} k!}\left(n + \frac{1}{2}\left(\frac{2p_1 p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{k-1}}{\alpha_k} p_k + \frac{\alpha_k}{1} p_{k+1}\right)\right)^k.$$

Thus, we have that if the result is correct for $m = k$ then it is correct for $m = k + 1$ when $\alpha_{k+1} = 1$. Now assume that $(p_1, p_2, \ldots, p_{k+1}) = \alpha_{k+1}$ (say).

If $\alpha_{k+1} \nmid n$, then $p^*(p_1, \ldots, p_{k+1}; n) = 0$. If $\alpha_{k+1} | n$, then

$$p^*(p_1, p_2, \ldots, p_{k+1}; n) = p^*\left(\frac{p_1}{\alpha_{k+1}}, \frac{p_2}{\alpha_{k+1}}, \ldots, \frac{p_{k+1}}{\alpha_{k+1}}; \frac{n}{\alpha_{k+1}}\right),$$

where $\left(\frac{p_1}{\alpha_{k+1}}, \ldots, \frac{p_{k+1}}{\alpha_{k+1}}\right) = 1$, and thus

$$p^*(p_1, \ldots, p_{k+1}; n) \leqslant \frac{1}{\frac{p_1}{\alpha_{k+1}} \cdots \frac{p_{k+1}}{\alpha_{k+1}} k!}\left(\frac{n}{\alpha_{k+1}} + \frac{1}{2}\left(\frac{\frac{2p_1}{\alpha_{k+1}} \cdot \frac{p_2}{\alpha_{k+1}}}{\frac{\alpha_2}{\alpha_{k+1}}} + \frac{\frac{\alpha_2}{\alpha_{k+1}}}{\frac{\alpha_3}{\alpha_{k+1}}} \cdot \frac{p_3}{\alpha_{k+1}} + \cdots + \frac{\frac{\alpha_{k-1}}{\alpha_{k+1}}}{\frac{\alpha_k}{\alpha_{k+1}}} \cdot \frac{p_k}{\alpha_{k+1}} + \frac{\frac{\alpha_k}{\alpha_{k+1}}}{\frac{\alpha_{k+1}}{\alpha_{k+1}}} \cdot \frac{p_{k+1}}{\alpha_{k+1}}\right)\right)^k$$

$$= \frac{\alpha_{k+1}}{p_1 p_2 \cdots p_{k+1} k!}\left(n + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3}p_3 + \cdots + \frac{\alpha_k}{\alpha_{k+1}}p_{k+1}\right)\right)^k.$$

Thus, the result is correct for $m = k + 1$ if it is correct for $m = k$. But, we know that the result is correct for $m = 3$, and hence the result is correct for $m \geqslant 3$. This completes the proof.

4. A COMPARISON WITH KNOWN PARTICULAR CASES

(a) The upper bound given by Rieger [6] is

$$p_m(n) < \frac{1}{m!(m-1)!}\left(n + \frac{m(m-3)}{4}\right)^{m-1} \quad \text{for } n \geqslant 0,\ m \geqslant 4.$$

We have $p_m(n) = p^*(1, 2, 3, \ldots, m; n - m)$ and $\alpha_2 = 1$; thus,

$$p_m(n) \leqslant \frac{1}{m!(m-1)!}\left(n - m + \frac{1}{2}(4 + 3 + 5 + \cdots + m)\right)^{m-1}.$$

$$\therefore \quad \text{Our result} = \frac{1}{m!(m-1)!}\left(n + \frac{m(m-3)}{4} + \frac{1}{2}\right)^{m-1} \quad \text{for } m \geqslant 3.$$

(b) H. Gupta [5] has given the following result for the particular case in which $p_1 = 1$:

$$\frac{\binom{n+m-1}{m-1}}{p_2p_3\cdots p_m} \leqslant p^*(1, p_2, p_3, \ldots, p_m; n) \leqslant \frac{\binom{n+p_2+p_3+\cdots+p_m}{m-1}}{p_2p_3\cdots p_m}$$

For the upper bound, we have

$$\binom{n+p_2+\cdots+p_m}{m-1} = \frac{(n+p_2+\cdots+p_m)!}{(m-1)!(n+p_2+\cdots+p_m-(m-1))!}$$

For large $n + p_2 + \cdots + p_m - (m - 1)$,

$$\sim \frac{1}{(m-1)!}\ \frac{(n+p_2+\cdots+p_m)^{n+p_2+\cdots+p_m+\frac{1}{2}} \cdot e^{-(m-1)}}{(n+p_2+\cdots+p_m-(m-1))^{n+p_2+\cdots+p_m+\frac{1}{2}-(m-1)}}$$

$$= \frac{(n+p_2+\cdots+p_m-(m-1))^{m-1}}{(m-1)!} \cdot \frac{e^{-(m-1)}}{\left(1 - \frac{(m-1)}{n+p_2+\cdots+p_m}\right)^{n+p_2+\cdots+p_m+\frac{1}{2}}}$$

$$\sim \frac{(n+p_2+\cdots+p_m-(m-1))^{m-1}}{(m-1)!}.$$

Thus, Gupta's result for the upper bound is asymptotic to

$$\frac{1}{1 \cdot p_2\cdots p_m(m-1)!}(n + p_2 + p_3 + \cdots + p_m - (m-1))^{m-1}.$$

Our result with $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 1$ as $p_1 = 1$ is sharper if

$$p_2 + p_3 + \cdots + p_m - (m - 1) > \frac{1}{2}(2p_2 + p_3 + \cdots + p_m)$$

or

$$p_3 + p_4 + \cdots + p_m > 2m - 2.$$

For arbitrarily large p_i, this is obviously satisfied as $\sum_{i=3}^{m} p_i$ will, in general, be much larger than $2m - 2$.

5. AN UPPER BOUND FOR $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$

This function represents the number of solutions of the inequality

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m \leqslant n$$

in integers $x_i \geqslant 0$ for $n \geqslant 0$. Alternatively, this represents the number of lattice points within and on the hypertetrahedron bounded by the planes $x_i = 0$ and the hyperplane

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n.$$

We can assume that $(p_1, p_2, \ldots, p_m) = 1$, and thus

$$\begin{aligned} &p^*(p_1, p_2, \ldots, p_m; \leqslant n) \\ &\leqslant \frac{1}{p_1p_2\cdots p_m(m-1)!} \sum_{r=0}^{n} \left(r + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{m-1}}{1} p_m\right)\right)^{m-1} \\ &\leqslant \frac{1}{p_1p_2\cdots p_m m!} \left(n + \frac{1}{2} + \frac{1}{2}\left(\frac{2p_1p_2}{\alpha_2} + \frac{\alpha_2}{\alpha_3} p_3 + \cdots + \frac{\alpha_{m-1}}{1} p_m\right)\right)^{m} \end{aligned}$$

for $n \geqslant 0$ and $m \geqslant 3$, using 2.2.

6. NUMERICAL RESULTS AND ASYMPTOTICS

Consider the example

$$p^*(60, 120, 150, 216, 243, 247; n),$$

where $\alpha_2 = 60$, $\alpha_3 = 30$, $\alpha_4 = 6$, $\alpha_5 = 3$, and $\alpha_6 = 1$. It is clear that α_{k+1} must divide α_k.

It is known [4] that if $(p_1, p_2, \ldots, p_m) = 1$ then $p^*(p_1, p_2, \ldots, p_m; n) > 0$ for sufficiently large n. This implies that there is a largest integer n for which $p^*(p_1, p_2, \ldots, p_m; n) = 0$. This greatest integer is denoted by

$$G(p_1, p_2, \ldots, p_m).$$

The paper [4] gives some upper bounds for $G(p_1, p_2, \ldots, p_m)$. Using these upper bounds and a numerical search, it can be found that

$$G(60, 120, 150, 216, 243, 247) = 1541.$$

For larger n, the partition function will be much smoother and the upper bound will become asymptotically better.

We have, for the previous particular numerical example, the following results.

n	$p^*(;\ n)$	Upper Bound	$p^*(;\leqslant n)$	Upper Bound
1541	0	136	7090	67396
6944	$.11723\times 10^5$	$.24412\times 10^5$	$.17050\times 10^8$	$.34057\times 10^8$
19760	$.19217\times 10^7$	$.25387\times 10^7$	$.68932\times 10^{10}$	$.89646\times 10^{10}$
39779	$.61270\times 10^8$	$.70673\times 10^8$	$.42470\times 10^{12}$	$.48535\times 10^{12}$
44505	$.11307\times 10^9$	$.12163\times 10^9$	$.82616\times 10^{12}$	$.93112\times 10^{12}$
60000	$.49311\times 10^9$	$.52036\times 10^9$	$.48728\times 10^{13}$	$.53274\times 10^{13}$
490000	$.16900\times 10^{14}$	$.17057\times 10^{14}$	$.13817\times 10^{19}$	$.13970\times 10^{19}$

CONCLUSION

An upper bound has been determined for $p^*(p_1, p_2, \ldots, p_m; n)$ and $p^*(p_1, p_2, \ldots, p_m; \leqslant n)$ for all $n \geqslant 0$ and $m \geqslant 3$.

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