A SYSTEMATIC SEARCH FOR UNITARY HYPERPERFECT NUMBERS

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1. INTRODUCTION

If m and t are natural numbers, we say that m is a unitary hyperperfect number of order t if

$$m = 1 + t[\sigma^*(m) - m - 1], \tag{1}$$

where $\sigma^*(m)$ denotes the sum of the unitary divisors of m. m is said to be a hyperperfect number of order t if

$$m = 1 + t[\sigma(m) - m - 1], \tag{2}$$

where σ is the usual divisor sum function. Hyperperfect numbers (HP's) were first studied by D. Minoli & R. Bear [4], while the study of unitary hyperperfect numbers (UHP's) was initiated by the present author [3]. H. J. J. te Riele [6] has found *all* (151) HP's less than 10^8 as well as many larger ones having more than two prime factors. D. Buell [2] has found all (146) UHP's less than 10^8. More recently, W. Beck & R. Najar [1] have studied the properties of HP's and UHP's. One of the results they obtained was the following.

Proposition 1: If m is a unitary hyperperfect number of order t, then $(m, t) = 1$ and m *and* t are of opposite parity.

The purpose of the present paper is to develop a search procedure, different from that employed by Buell, which can be used to find all of the unitary hyperperfect numbers less than a specified bound with a specified number of distinct prime factors (provided the necessary computer time is available).

2. THE GENERAL PROCEDURE

Suppose that $m = ar^{\gamma}s^{\lambda}$, where r and s are distinct primes, $\gamma\lambda \neq 0$, and $(a, rs) = 1$. If m is a unitary hyperperfect number of order t, then, since σ^* is multiplicative and $\sigma^*(r^{\gamma}) = 1 + r^{\gamma}$, it follows from (1) that

$$[a - t(\sigma^*(a) - a)]r^{\gamma}s^{\lambda} - t\sigma^*(a)[r^{\gamma} + s^{\lambda}] = 1 + t[\sigma^*(a) - 1].$$

Multiplying this equality by $a - t(\sigma^*(a) - a)$ and then adding $[t\sigma^*(a)]^2$ to each side, we obtain

$$\begin{aligned}&\{[a - t(\sigma^*(a) - a)]r^{\gamma} - t\sigma^*(a)\}\{[a - t(\sigma^*(a) - a)]s^{\lambda} - t\sigma^*(a)\}\\ &= [a - t(\sigma^*(a) - a)][1 + t(\sigma^*(a) - 1)] + [t\sigma^*(a)]^2.\end{aligned} \tag{3}$$

If AB, where $1 \leqslant A < B$, is the "correct" factorization of the right-hand member of (3), then we see that

$$r^\gamma = [t\sigma^*(a) + A]/[a - t(\sigma^*(a) - a)],$$
$$s^\lambda = [t\sigma^*(a) + B]/[a - t(\sigma^*(a) - a)]. \tag{4}$$

Since the steps just described are reversible, given values of a and t, *if* a factorization AB of the right-hand member of (3) can be found for which the right-hand members of (4) are distinct prime powers relatively prime to a, then the integer $ar^\gamma s^\lambda$ is a unitary hyperperfect number of order t. Of course, for most values of a and t the right-hand members of (4) will not both be integers, let alone prime powers. It should be mentioned that the above derivation of (4) is basically due to Euler via H. J. J. te Riele (see [5]).

3. THE CASE $a = 1$

If, in (4), we set $a = 1$, then, since $\sigma^*(1) = 1$, it follows that $r^\gamma = t + A$ and $s^\lambda = t + B$, where, from (3), $AB = 1 + t^2$. Suppose that t is odd. Then $AB \equiv 2 \pmod 8$ and it follows that A and B are of opposite parity. Therefore, without loss of generality, $r = 2$ and, since $3 | ts^\lambda$ (see Fact 1 in [3]), we have proved the following result.

Proposition 2: If $m = r^\gamma s^\lambda$ is a unitary hyperperfect number of odd order t, then $2 | m$ and either $m = 2^\gamma 3^\lambda$ or $3 | t$.

Using the CDC CYBER 750 at the Temple University Computing Center, a search was made for all unitary hyperperfect numbers less than 10^{14} of the form $2^\gamma 3^\lambda$. Only two were found:

$$2 \cdot 3 \ (t = 1) \quad \text{and} \quad 2^5 \cdot 3^2 \ (t = 7).$$

The search required less than one second.

We now drop the restriction that t be odd.

Proposition 3: If $m = r^\gamma s^\lambda = RS$ is a unitary hyperperfect number of order t, then $m > 4t^2$.

Proof: $RS = 1 + t(\sigma^*(RS) - RS - 1) = 1 + t(R + S)$. Therefore, $R > t(1 + R/S)$. Similarly, $S > t(1 + S/R)$, and it follows that

$$RS > t^2(1 + R/S + S/R + 1) > 4t^2.$$

From Proposition 3, we see that all unitary hyperperfect numbers less than 10^{10} and of the form $r^\gamma s^\lambda$ can be found by decomposing $1 + t^2$, for $1 \leqslant t < 50000$, into two factors A and B and then testing $t + A$ and $t + B$ to see if each is a prime power. This was done, and 822 UHP's less than 10^{10} with two components were found. 790 were square-free and, therefore, also HP's. Of the remaining 32 "pure" UHP's, all but one, $3^2 \cdot 2^5$ $(t = 7)$, were of the form $r^\gamma s$ or rs^λ. t was odd for only ten of the 822 numbers, the two largest being

$$2^{13} \cdot 33413 \ (t = 6579) \quad \text{and} \quad 2^{15} \cdot 238037 \ (t = 28803).$$

The complete search took about five minutes of computer time.

4. AN IMPORTANT INEQUALITY

In this section, we shall generalize the inequality of Proposition 3.

Proposition 4: Suppose that m is a unitary hyperperfect (*or* a hyperperfect) number of order t with exactly n prime-power components. Then $m > (nt)^n$.

Proof: Suppose first that $n = 3$ and $m = p^\alpha q^\beta r^\gamma = PQR$, where $P > Q > R$. From (1) [and (2)], it follows easily that

$$PQR > t(PQ + PR + QR).$$

If $A = P/Q$ and $B = P/R$, then

$$P > t(1 + A + B),\quad Q > t(1 + B/A + 1/A),\text{ and } R > t(1 + A/B + 1/B).$$

Therefore,

$$m = PQR > t^3(1 + A + B)^3/AB. \tag{5}$$

If $F(x, y) = (1 + x + y)^3/xy$, where $x > 0$ and $y > 0$, then

$$\partial F/\partial x = (1 + x + y)^2(2x - y - 1)/x^2y$$

and

$$\partial F/\partial y = (1 + x + y)^2(2y - x - 1)/xy^2.$$

It follows easily that, if $x > 0$ and $y > 0$, then $F(x, y) \geqslant F(1, 1) = 3^3$. From (5), we have $m > (3t)^3$.

Now suppose that $n = 4$ and $m = p^\alpha q^\beta r^\gamma s^\lambda = PQRS$, where $P > Q > R > S$. From (1) [or (2)],

$$PQRS > t(PQR + PQS + PRS + QRS).$$

If $A = P/Q$, $B = P/R$, and $C = P/S$, then

$$P > t(1 + A + B + C),\quad Q > t(1 + B/A + C/A + 1/A),$$

$$R > t(1 + A/B + C/B + 1/B),\text{ and } S > t(1 + A/C + B/C + 1/C).$$

Therefore,

$$m = PQRS > t^4(1 + A + B + C)^4/ABC. \tag{6}$$

If $G(x, y, z) = (1 + x + y + z)^4/xyz$, where $x > 0$, $y > 0$, $z > 0$, then

$$\partial G/\partial x = (1 + x + y + z)^3(3x - y - z - 1)/x^2yz,$$

$$\partial G/\partial y = (1 + x + y + z)^3(3y - x - z - 1)/y^2xz,$$

and

$$\partial G/\partial z = (1 + x + y + z)^3(3z - x - y - 1)/z^2xy.$$

It follows that $G(x, y, z)$ has a minimum at $(1, 1, 1)$ *and* that $G(x, y, z) \geqslant 4^4$ if $x > 0$, $y > 0$, $z > 0$. From (6), we see that $m > (4t)^4$.

A similar argument can be used for any value of n that exceeds 4.

5. THE CASE $a = p^\alpha$

If, in (4) and the right-hand member of (3), we set $a = p^\alpha$, then, since $\sigma^*(p^\alpha) = p^\alpha + 1$, it follows that

$$r^\gamma = (t\sigma^*(p^\alpha) + A)/(p^\alpha - t) \quad \text{and} \quad s^\lambda = (t\sigma^*(p^\alpha) + B)/(p^\alpha - t) \tag{7}$$

where

$$AB = (p^\alpha - t)(1 + tp^\alpha) + t^2(p^\alpha + 1)^2. \tag{8}$$

If $m = p^\alpha r^\gamma s^\lambda$ is a UHP of order t such that $m < 10^9$, then it is easy to see that if p^α is the smallest prime-power component of m, $p^\alpha < 1000$. From Proposition 4, $t < 1000/3$. All solutions of (7) and (8) (with $A < B$) were sought with $2 \leqslant p^\alpha \leqslant 997$, $1 \leqslant t \leqslant 333$, and $p^\alpha r^\gamma s^\lambda < 10^9$. The search yielded nine UHP's less than 10^9. Five of these were given in [2]. The four new ones are:

$2^6 \cdot 659 \cdot 2693$ ($t = 57$); $67 \cdot 643 \cdot 79^2$ ($t = 60$);

$547 \cdot 569 \cdot 1259$ ($t = 228$); $7^2 \cdot 79 \cdot 119971$ ($t = 30$).

The search required about thirty minutes of computer time.

6. THE UHP's LESS THAN 10^9

Let M_n denote the set of all unitary hyperperfect numbers m such that $m < 10^9$ and m has exactly n distinct prime divisors. From Fact 2 in [3], M_1 is empty and, from the searches described in Sections 3 and 5, M_2 and M_3 have 330 and 9 elements, respectively. Since $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 > 10^9$, we see that M_n is empty if $n > 9$. If $n = 8$ or 9, then, from Proposition 4, it follows easily that $t = 1$ so that, if $m \in M_8$ or $m \in M_9$, then m is a unitary perfect number ($\sigma^*(m) = 2m$). Since there are no unitary perfect numbers less than 10^9 with 8 or 9 prime-power components (see [7]), it follows that both M_8 and M_9 are empty.

If $m < 10^9$, then, from Proposition 4, if $n = 4$, then $t \leqslant 44$, if $n = 5$, then $t \leqslant 12$, if $n = 6$, then $t \leqslant 5$, if $n = 7$, then $t \leqslant 2$. Subject to these restrictions on t, and with a restricted so that r^γ is greater than every prime-power component of a while $ar^\gamma s^\lambda < 10^9$, a search was made for solutions of (4). This search required two-and-one-half hours of computer time, and it was found that M_4, M_6, and M_7 are empty, while M_5 has one element, $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ ($t = 1$). Thus, there are exactly 340 UHP's less than 10^9.

It should, perhaps, be mentioned that while M_4 is empty, one UHP with four prime-power components *was* found: $59 \cdot 149 \cdot 29077 \cdot 10991483959$ ($t = 42$) is both a UHP and an HP (since it is square free). It does not appear in te Riele's lists of HP's and may be the smallest HP with exactly four distinct prime factors.

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