A COMPLETE CHARACTERIZATION OF B-POWER FRACTIONS THAT CAN BE REPRESENTED AS SERIES OF GENERAL n-BONACCI NUMBERS

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(Submitted April 1985)

1. INTRODUCTION AND MAIN RESULT

In 1953, Fenton Stancliff [5] noted that

$$\frac{1}{89} = .0112358\ldots = \sum_{k=0}^{\infty} 10^{-(k+1)} F_k,$$

$$\begin{array}{l} .0112358 \\ \quad\;\; 13 \\ \quad\;\;\; 21 \\ \quad\;\;\; \ldots \end{array}$$

where F_k denotes the k^{th} Fibonacci number. Until recently, this expansion was regarded as an anomalous numerical curiosity, possibly related to the fact that 89 is a Fibonacci number (see Remark in [5]), but not generalizing to other fractions in an obvious manner.

In 1980, C. F. Winans [6] showed that the sums $\sum 10^{-(k+1)} F_{\alpha k}$ approximate 1/71, 2/59, and 3/31 for $\alpha = 2, 3$, and 4, respectively. Moreover, he showed that the sums $\sum 10^{-2(k+1)} F_{\alpha k}$ approximate 1/9899, 1/9701, 2/9599, and 3/9301 for $\alpha = 1, 2, 3$, and 4, respectively.

Since then, several authors proved general theorems on fractions that can be represented as series involving Fibonacci numbers and general n-Bonacci numbers [1, 2, 3, 4]. In the present paper we will prove a theorem which includes as special cases all the earlier results. We introduce some notation in order to state our theorem.

Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, W_0, W_1, \ldots, W_m$, and B be given. Construct the sequence W_k by the recursion

$$W_{n+m+1} = \sum_{r=0}^{m} A_r W_{n+m-r}$$

for $n \geqslant 0$ or, equivalently, by the formula

$$W_n = \sum_{r=0}^{m} \lambda_r \omega_r^n$$

for any integer n where ω_r ($r = 0, 1, \ldots, m$) are the zeros of the polynomial

$$q(z) = z^{m+1} - \sum_{r=0}^{m} A_r z^{m-r}$$

and $(\lambda_0, \lambda_1, \ldots, \lambda_m)$ is the unique solution of the system of $m + 1$ linear equations

$$\sum_{r=0}^{m} \lambda_r \omega_r^n = W_n \qquad (n = 0, 1, \ldots, m)$$

(see [2], p. 35). Finally, we introduce, for any integer α,

$$M(m) = \prod_{r=0}^{m} (B - \omega_r^{\alpha}).$$

Theorem: For integers $\alpha \geqslant 1$, $\beta \geqslant 0$, and any complex B satisfying

$$\max_{0 \leqslant r \leqslant m} |\omega_r^{\alpha}/B| < 1,$$

we have the formula

$$M(m) \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta} = B \cdot \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \prod_{\substack{0 \leqslant k \leqslant m \\ k \neq r}} (B - \omega_k^{\alpha}).$$

Remark: In the above formula, $M(m)$ and the right-hand side are in fact integers if $B, A_0, A_1, \ldots, A_m, W_0, W_1, \ldots, W_m$ are all integers.

Now we can comment on earlier results in more detail. In 1981, Hudson and Winans [1] handled the case of the ordinary Fibonacci sequence with $\beta = 0$, $B = 10^n$. According to [3] and [4], their result can be written as

$$\sum_{k=1}^{\infty} 10^{-n(k+1)} F_{\alpha k} = \frac{F_{\alpha}}{10^{2n} - 10^n L_{\alpha} - (-1)^{\alpha}},$$

where L_{α} denote the Lucas numbers. Also in 1981, Long [4] treated the case of the general Fibonacci sequence, i.e., $m = 1$ and arbitrary A_0, A_1, W_0, W_1, and B, with the restriction, however, to $\alpha = 1$, $\beta = 0$. In 1985, Köhler [2] gave the generalization for arbitrary $m, A_0, A_1, \ldots, A_m, W_0, W_1, \ldots, W_m, B$, again with the restriction to $\alpha = 1$, $\beta = 0$. His result is

$$\sum_{k=1}^{\infty} B^{-k} W_{k-1} = p(B)/q(B),$$

where p is a polynomial of degree m with explicitly given coefficients. Also in 1985, Lee [3] discussed the cases $m = 1$ and $m = 2$ of general Fibonacci and Tribonacci sequences with arbitrary α and β. The results of [3] will be deduced from our Theorem in Examples 1 and 2 below. For this purpose, we introduce the notation

$$S_n = \sum_{r=0}^{m} \omega_r^n, \quad L(m) = M(m) \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta}.$$

Proof of Theorem: We have

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k + \beta} = \sum_{k=0}^{\infty} B^{-k} \sum_{r=0}^{m} \lambda_r \omega_r^{\alpha k + \beta}$$

$$= \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \left(\sum_{k=0}^{\infty} (B^{-1} \omega_r^{\alpha})^k \right) = B \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \frac{1}{B - \omega_r^{\alpha}}.$$

Convergence is guaranteed by the condition on B. In the same way, we obtain

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = \sum_{r=0}^{m} \lambda_r \omega_r^{\beta} \cdot \frac{\omega_r^{\alpha}}{B - \omega_r^{\alpha}}.$$

Multiplying with $M(m)$ yields the Theorem.

Remark: Partial sums of the series in our Theorem can be expressed by these series, according to the formula

$$B^n \cdot \sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B^n \cdot \sum_{k=0}^{n} B^{-k} W_{\alpha k+\beta} + \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+(\alpha n+\beta)}.$$

2. EXAMPLES

Example 1: The general Fibonacci sequence. Take $m = 1$. Then we have

$$W_{n+2} = A_0 W_{n+1} + A_1 W_n,$$

$$M(1) = (B - \omega_0^{\alpha})(B - \omega_1^{\alpha}) = B^2 - BS_{\alpha} + (-A_1)^{\alpha},$$

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = \{\lambda_0 \omega_0^{\alpha+\beta}(B - \omega_1^{\alpha}) + \lambda_1 \omega_1^{\alpha+\beta}(B - \omega_0^{\alpha})\}/M(1)$$
$$= (BW_{\alpha+\beta} - (-A_1)^{\alpha} W_{\beta})/M(1),$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B(BW_{\beta} - (-A_1)^{\alpha} W_{\beta-\alpha})/M(1).$$

As to the partial sums, we get

$$\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} = B^n L(1)/M(1) - \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\alpha n+\beta}$$
$$= \frac{B^n L(1) - BW_{\alpha(n+1)+\beta} + (-A_1)^{\alpha} W_{\alpha n+\beta}}{B^2 - BS_{\alpha} + (-A_1)^{\alpha}}.$$

These formulas are equal to Theorems 1-3 of [3].

Example 2: The general Tribonacci sequence. Take $m = 2$. Then we obtain

$$W_{n+3} = A_0 W_{n+2} + A_1 W_{n+1} + A_2 W_n,$$

$$M(2) = B^3 - B^2 S_{\alpha} + BA_2^{\alpha} S_{-\alpha} - A_2^{\alpha},$$

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = (B^2 W_{\alpha+\beta} + B(W_{2\alpha+\beta} - S_{\alpha} W_{\alpha+\beta}) + A_2^{\alpha} W_{\beta})/M(2),$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B(B^2 W_{\beta} + B(W_{\alpha+\beta} - S_{\alpha} W_{\beta}) + A_2^{\alpha} W_{\beta-\alpha})/M(2),$$

$$\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} = B^n L(2)/M(2) - \sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\alpha n+\beta}$$

(continued)

$$= \frac{B^n L(2) - B^2 W_{\alpha(n+1)+\beta} + B(S_\alpha W_{\alpha(n+1)+\beta} - W_{\alpha(n+2)+\beta}) - A_2^\alpha W_{\alpha n+\beta}}{B^3 - B^2 S_\alpha + BA_2^\alpha S_{-\alpha} - A_2^\alpha}.$$

These formulas are equal to (9) and Theorems 7 and 8 in [3], and a misprint in Theorem 7 in [3] is corrected.

Example 3: The general Tetranacci sequence. Take $m = 3$. Then we have

$$W_{n+4} = A_0 W_{n+3} + A_1 W_{n+2} + A_2 W_{n+1} + A_3 W_n,$$

$$M(3) = B^4 - B^3 S_\alpha + B^2(S_\alpha^2 - S_{2\alpha})/2 - B(-A_3)^\alpha S_{-\alpha} + (-A_3)^\alpha,$$

$$\sum_{k=1}^{\infty} B^{-k} W_{\alpha k+\beta} = \{B^3 W_{\alpha+\beta} + B^2(W_{2\alpha+\beta} - S_\alpha W_{\alpha+\beta}) + B(-A_3)^\alpha (S_{-\alpha} W_\beta - W_{\beta-\alpha}) - (-A_3)^\alpha W_\beta\}/M(3),$$

$$\sum_{k=0}^{\infty} B^{-k} W_{\alpha k+\beta} = B\{B^3 W_\beta + B^2(W_{\alpha+\beta} - S_\alpha W_\beta) + B(2W_{2\alpha+\beta} - 2S_\alpha W_{\alpha+\beta} + (S_\alpha^2 - S_{2\alpha})W_\beta)/2 - (-A_3)^\alpha W_{\beta-\alpha}\}/M(3),$$

$$\sum_{k=0}^{n} B^{n-k} W_{\alpha k+\beta} = \{B^n L(3) - B^3 W_{\alpha(n+1)+\beta} - B^2(W_{\alpha(n+2)+\beta} - S_\alpha W_{\alpha(n+1)+\beta}) - B(-A_3)^\alpha (S_{-\alpha} W_{\alpha n+\beta} - W_{\alpha(n+1)+\beta}) + (-A_3)^\alpha W_{\alpha n+\beta}\}/M(3).$$

Formulas for $m \geqslant 4$ can be obtained in a similar manner.

ACKNOWLEDGMENTS

We are deeply thankful to Dr. Hong-Jinh Chang and to the referee for the earnest discussions and valuable suggestions made.

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