ORDINARY GENERATING FUNCTIONS FOR PELL POLYNOMIALS

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1. INTRODUCTION

The object of this paper is to investigate, by using a variety of methods, the properties of Pell polynomials $P_n(x)$ and the Pell-Lucas polynomials $Q_n(x)$ [6] which are derivable from their generating functions. Brief acquaintance with the main aspects of [6] is desirable.

In an endeavor to conserve space, we will generally offer only an indication of the potential development, with a minimum of results, so that just a representative sample of the material available is presented. Omitted information will be happily supplied on request. Among the many facets of this exposition, we find the sections numbered 4 and 5 especially appealing.

For visual convenience, the functional notation will be suppressed and an abbreviated notation used, e.g., $P_n(x) \equiv P_n$, $Q_n(x) \equiv Q_n$.

First, we introduce the notation

$$P(j,\ m,\ k,\ x,\ y) = \sum_{r=0}^{\infty} P_{mr+k}^{j} y^r, \tag{1.1}$$

$$Q(j,\ m,\ k,\ x,\ y) = \sum_{r=0}^{\infty} Q_{mr+k}^{j} y^r. \tag{1.2}$$

Then, for example, by difference equations [6],

$$P(1,\ 1,\ 0,\ x,\ y) = y\Delta \tag{1.3}$$

or, equivalently,

$$P(1,\ 1,\ 1,\ x,\ y) = \Delta = \sum_{r=0}^{\infty} P_{r+1} y^r, \tag{1.4}$$

and

$$Q(1,\ 1,\ 0,\ x,\ y) = (2 - 2xy)\Delta \tag{1.5}$$

or, equivalently,

$$Q(1,\ 1,\ 1,\ x,\ y) = (2x + 2y)\Delta = \sum_{r=0}^{\infty} Q_{r+1} y^r, \tag{1.6}$$

in all of which

$$\Delta = (1 - 2xy - y^2)^{-1} = \Delta(x,\ y,\ 1,\ 1) \quad [\text{cf. } (1.8)]. \tag{1.7}$$

Result (1.4), for example, may also be obtained using the method of column generators [1] with the aid of binomial coefficient expressions for P_n given in [7]. Matrices and Binet forms may also be utilized (see [7]) in establishing (1.3)-(1.6).

Let us introduce the symbolism

$$\Delta_{(m)}^{(j)} \equiv \Delta(x, y, j, m)$$

[cf. (1.13)] in which the superscript and subscript will be suppressed when $j = 1$ and/or $m = 1$, e.g., $\Delta_{(1)}^{(1)} = \Delta$ [cf. (1.7)] and

$$\Delta_{(m)} = (1 - Q_m y + (-1)^m y^2)^{-1} \equiv \Delta(x, y, 1, m) \tag{1.8}$$

whence (1.7) follows when $m = 1$. Replacing y by $-y$, we write

$$\Delta'_{(m)} \equiv \Delta(x, -y, 1, m). \tag{1.9}$$

Furthermore, with $m = 1$, let

$$\Delta^{(j)} \equiv \Delta(x, y, j, 1) = \left[\sum_{r=0}^{j+1} (-1)^{\frac{r(r+1)}{2}} \{j + 1, r\} y^r\right]^{-1}, \tag{1.10}$$

where the symbol $\{a, b\}$, defined in [8], is

$$\{a, b\} = \prod_{i=1}^{a} P_i \Big/ \left(\prod_{i=1}^{b} P_i\right)\left(\prod_{i=1}^{a-b} P_i\right) \tag{1.11}$$

Thus, in particular, from (1.10) and (1.11),

$$\begin{cases} \Delta = (1 - P_2 y - y^2)^{-1} & \text{as in (1.7)} \\ \Delta^{(2)} = (1 - P_3 y - P_3 y^2 + y^3)^{-1} \\ \Delta^{(3)} = (1 - P_4 y - (P_3 P_4 / P_1 P_2) y^2 + P_4 y^3 + y^4)^{-1} \end{cases} \tag{1.12}$$

More generally,

$$\Delta_{(m)}^{(j)} = \left[\sum_{r=0}^{j+1} (-1)^{\frac{r[m(r-1)+2]}{2}} \{j + 1, r\}_m y^r\right]^{-1}, \tag{1.13}$$

in which

$$\{a, b\}_m = \prod_{i=1}^{a} P_{im} \Big/ \left(\prod_{i=1}^{b} P_{im}\right)\left(\prod_{i=1}^{a-b} P_{im}\right). \tag{1.14}$$

The case $j = 1$ occurs in (1.8), while the case $m = 1$ occurs in (1.10). Later, in (6.6), we refer to the case $j = 3$, i.e., to $\Delta_{(m)}^{(3)}$.

Some useful results from [7] are collected here for later reference:

$$Q_{n+r} + Q_{n-r} = \begin{cases} Q_n Q_r & r \text{ even,} \\ 4(x^2 + 1) P_n P_r & r \text{ odd.} \end{cases} \tag{1.15}$$

$$P_{n+1}^2 - (4x^2 + 2) P_n^2 + P_{n-1}^2 = 2(-1)^n. \tag{1.16}$$

$$P_{m(r+1)+k} - Q_m P_{mr+k} + (-1)^m P_{m(r-1)+k} = 0. \tag{1.17}$$

Also important for our matrix treatment are (see [6]):

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}; \tag{1.18}$$

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \quad \text{so } |P^n| = (-1)^n. \tag{1.19}$$

Consult [6], [7], and [8] for details of some of the applications of P.

2. APPLICATIONS OF GENERATING FUNCTIONS

Using (1.17) as a difference equation, we find eventually that

$$P(1, m, k, x, y) = [P_k + (-1)^k P_{m-k} y]\Delta_{(m)}. \tag{2.1}$$

Similarly,

$$Q(1, m, k, x, y) = [Q_k + (-1)^{k-1} Q_{m-k} y]\Delta_{(m)}. \tag{2.2}$$

The specializations given in (1.3) and (1.5) follow immediately. Numerous other specializations of some interest, e.g., those for

$$P(1, 2, 0, x, y),\ P(1, 2, 1, x, y),\ P(1, 3, 3, x, -y)$$

and $\quad Q(1, 2, 1, x, -y)$,

are listed in [7].

Differentiating (1.4) with respect to y, we obtain

$$\sum_{r=0}^{\infty} rP_{r+1} y^{r-1} = (2x + 2y)\Delta^2. \tag{2.3}$$

Likewise

$$\sum_{r=0}^{\infty} rQ_{r+1} y^{r-1} = [4x^2 + 2 + 4xy + 2y^2]\Delta^2. \tag{2.4}$$

Replacing y by $-y$ gives generating functions of some importance. Results (2.3) and (2.4) may be extended if we differentiate (2.1) and (2.2) w.r.t. y, but the process is somewhat algebraically messy.

Now, (2.3) leads to an interesting summation. With (1.4) and (1.6) it gives

$$\sum_{r=0}^{\infty} rP_{r+1} y^{r-1} = \left\{ \sum_{r=0}^{\infty} P_{r+1} y^n \right\} \left\{ \sum_{r=0}^{\infty} Q_{r+1} y^n \right\}. \tag{2.5}$$

Equate coefficients of y^r on both sides, thus obtaining

$$(r + 1)P_{r+2} = \sum_{r=0}^{r+1} P_i Q_{r+2-i}. \tag{2.6}$$

Next, differentiate (1.5) w.r.t. y. Then

$$\sum_{r=0}^{\infty} (r + 1)Q_{r+1} y^r = (2x + 4y - 2xy^2)\Delta^2. \tag{2.7}$$

Combining (1.4) and (2.7), we find

$$\sum_{r=0}^{\infty}(r+1)Q_{r+1}y^r - \sum_{r=0}^{\infty}(r+1)\ P_{r+2}y^r = y(2-2xy)\Delta^2 \tag{2.8}$$
$$= \left\{\sum_{r=0}^{\infty}P_ry^r\right\}\left\{\sum_{r=0}^{\infty}Q_ry^r\right\}$$

by (1.3) and (1.5).

Equate coefficients to get

$$(r+1)(Q_{r+1} - P_{r+2}) = \sum_{i=0}^{r}P_iQ_{r-i}. \tag{2.9}$$

Differentiating in (1.3) w.r.t. y, then multiplying by y, we determine a generating function for rP_r, namely,

$$\sum_{r=0}^{\infty}rP_ry^r = y(1+y^2)\Delta^2. \tag{2.10}$$

Similarly,

$$\sum_{r=0}^{\infty}rQ_ry^r = (2xy + 4y^2 - 2xy^3)\Delta^2. \tag{2.11}$$

Generating functions may be used to derive already known properties of Pell polynomials, e.g.,

$$\sum_{n=0}^{\infty}Q_ny^n = (2-2xy)\Delta \qquad \text{by (1.5)}$$
$$= \Delta + (1-2xy)\Delta$$
$$= \sum_{n=0}^{\infty}P_{n+1}y^n + \sum_{n=0}^{\infty}P_{n-1}y^n \quad \text{by (1.4) and (2.1)},$$

whence $\quad Q_n = P_{n+1} + P_{n-1} \qquad$ [6, equation (2.1)].

Moreover, we may show that

$$Q(1, 1, 1, x, y) + Q(1, 1, -1, x, y) = 4(x^2+1)P(1, 1, 0, x, y),$$

whence $\quad Q_{n+1} + Q_{n-1} = 4(x^2+1)P_n \quad$ [cf. (1.15)].

New, but less elementary, identities may also be established. For instance,

$$\sum_{n=0}^{\infty}\{Q_nQ_{m-1} + Q_{n+1}Q_m\}y^n$$
$$= [(2-2xy)Q_{m-1} + (2x+2y)Q_m]\Delta \qquad \text{by (1.5) and (1.6)}$$
$$= [(2xQ_m + 2Q_{m-1}) + (2Q_m - 2xQ_{m-1})y]\Delta$$
$$= 4(x^2+1)(P_m + yP_{m-1})\Delta$$

by (1.13) and the recurrence relation for Q_m.

Terms in y^n being equated, we derive

$$Q_nQ_{m-1} + Q_{n+1}Q_m = 4(x^2+1)P_{m+n}. \tag{2.12}$$

Following the technique of Serkland [9] for Pell numbers, we can also establish fresh identities involving Pell polynomials. See [7] for details. A representative result incorporating this process is

$$P_u P_v P_w = \sum_{k=0}^{w-1} \left\{ P_{u+v+w-k} P_{k+1} - P_{u+k+1} P_{v+w-k} \right\}. \tag{2.13}$$

Finite series may be summed using a generating function. To illustrate this contention, choose

$$\begin{aligned} \sum_{r=1}^{m} P_r y^r &= \sum_{r=0}^{\infty} P_r y^r - \sum_{r=0}^{\infty} P_{r+m+1} y^r \\ &= y\{1 - (P_{m+1} + yP_m)\}\Delta \qquad \text{by (1.3) and (2.1).} \end{aligned}$$

Then, $y = 1$ gives equation (2.11) in [6].

Ideas of Hoggatt [2] in relation to Fibonacci and Lucas numbers may be extended to generators of Pell polynomials. For example,

$$\begin{aligned} &\sum_{k=0}^{\infty} 4^k (x^2 + 1)^k P_{2k+1} y^{2k+1} \qquad (2.14) \\ &= yP(1, 2, 1, x, 4(x^2 + 1)y^2) \qquad \text{by (1.1)} \\ &= y^2\{1 - 4(x^2 + 1)y^2\}\delta_{(2)} \qquad \text{by (2.1)} \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=0}^{\infty} 4^k (x^2 + 1)^k Q_{2k+2} y^{2k+2} \qquad (2.15) \\ &= y^2 Q(1, 2, 2, x, 4(x^2 + 1)y^2) \qquad \text{by (1.2)} \\ &= y^2\{(4x^2 + 2) - 8y^2(x^2 + 1)\}\delta_{(2)} \qquad \text{by (2.2)} \end{aligned}$$

where, in (2.14) and (2.15), $\delta_{(2)}$ means $\Delta_{(2)}$ with y replaced by $4(x^2 + 1)y$ [cf. (1.8)].

Add (2.14) and (2.15). Simplifying, we are left with

$$\begin{aligned} &\sum_{k=0}^{\infty} 4^k (x^2 + 1)^k \{P_{2k+1} + yQ_{2k+2}\} y^{2k+1} \qquad (2.16) \\ &= \frac{y - 2y^2}{1 - 4(x^2 + 1)y + 4(x^2 + 1)y^2} \end{aligned}$$

Further details appear in [7].

3. ELEMENTARY RELATIONS AMONG GENERATING FUNCTIONS

Analogous relations to those among polynomials may be determined for generating functions. Consider, for instance, the derivation of the recurrence relation

$$\begin{aligned} P(1,\ 1,\ n+2,\ x,\ y) &= (P_{n+2} + yP_{n+1})\Delta \quad \text{by (2.1)} \qquad (3.1)\\ &= (2x\{P_{n+1} + yP_n\} + P_n + yP_{n-1})\Delta \quad \text{by the definition of } P_n\\ &= 2xP(1,\ 1,\ n+1,\ x,\ y) + P(1,\ 1,\ n,\ x,\ y) \text{ by (2.1).} \end{aligned}$$

Likewise,

$$Q(1,\ 1,\ n+2,\ x,\ y) = 2xQ(1,\ 1,\ n+1,\ x,\ y) + Q(1,\ 1,\ n,\ x,\ y). \qquad (3.2)$$

It might be noted that the direct generating function analogue of

$$Q_n = P_{n+1} + P_{n-1}$$

flows almost immediately from (2.1) and (2.2).

Matrix representations of the generating functions are, in the notation of [8] for the matrix P,

$$\begin{bmatrix} P(1,\ 1,\ n,\ x,\ y) \\ P(1,\ 1,\ n-1,\ x,\ y) \end{bmatrix} = P^{n-1} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \qquad (3.3)$$

$$\begin{bmatrix} Q(1,\ 1,\ n,\ x,\ y) \\ Q(1,\ 1,\ n-1,\ x,\ y) \end{bmatrix} = P^{n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \qquad (3.4)$$

$$P(1,\ 1,\ n,\ x,\ y) = [1 \quad 0]P^{n-1} \begin{bmatrix} P(1,\ 1,\ 1,\ x,\ y) \\ P(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}, \qquad (3.5)$$

$$Q(1,\ 1,\ n,\ x,\ y) = [1 \quad 0]P^{n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix}. \qquad (3.6)$$

Now let us apply these matrices to obtain formulas for Pell and Pell-Lucas generating functions. First,

$$\begin{aligned} Q(1,\ 1,\ m+n,\ x,\ y) &= [1 \quad 0]P^{m+n-1} \begin{bmatrix} Q(1,\ 1,\ 1,\ x,\ y) \\ Q(1,\ 1,\ 0,\ x,\ y) \end{bmatrix} \quad \text{by (3.6)} \qquad (3.7)\\ &= [P_m \quad P_{m-1}] \begin{bmatrix} Q(1,\ 1,\ n+1,\ x,\ y) \\ Q(1,\ 1,\ n,\ x,\ y) \end{bmatrix} \quad \text{by (3.4) and (1.19)}\\ &= P_mQ(1,\ 1,\ n+1,\ x,\ y) + P_{m-1}Q(1,\ 1,\ n,\ x,\ y). \end{aligned}$$

A similar formula pertains to $P(1,\ 1,\ m+n,\ x,\ y)$, viz.,

$$P(1,1,m+n,x,y) = P_mP(1,\ 1,\ n+1,\ x,\ y) + P_{m-1}P(1,\ 1,\ n,\ x,\ y). \qquad (3.8)$$

Of course, (3.1) and (3.2) are special cases of (3.7) and (3.8) when $m = 2$.

Representative of another set of results is

$$P(1, 1, m+n, x, y) + (-1)^n P(1, 1, m-n, x, y) = Q_n P(1, 1, m, x, y) \tag{3.9}$$

Analogues of *Simson's formulas* can be established. Thus,

$$P^2(1, 1, n, x, y) - P(1, 1, n+1, x, y)P(1, 1, n-1, x, y) \tag{3.10}$$

$$= \begin{vmatrix} P(1, 1, n, x, y) & P(1, 1, n+1, x, y) \\ P(1, 1, n-1, x, y) & P(1, 1, n, x, y) \end{vmatrix}$$

$$= \left| P^{n-1}\begin{bmatrix} P(1, 1, 1, x, y) \\ P(1, 1, 0, x, y) \end{bmatrix} \vdots\ P^{n}\begin{bmatrix} P(1, 1, 1, x, y) \\ P(1, 1, 0, x, y) \end{bmatrix} \right| \quad \text{by (3.3)}$$

$$= |P^{n-1}| \begin{vmatrix} P(1, 1, 1, x, y) & 2xP(1, 1, 1, x, y) + P(1, 1, 0, x, y) \\ P(1, 1, 0, x, y) & P(1, 1, 1, x, y) \end{vmatrix} \quad \text{by (1.18)}$$

$$= (-1)^{n-1}\{P^2(1, 1, 1, x, y) - P(1, 1, 2, x, y)P(1, 1, 0, x, y)\} \quad \text{by (3.1)}$$

$$= (-1)^{n-1}(1 - 2xy - y^2)\Delta^2 \quad \text{by (1.3), (1.4), and (2.1)}$$

$$= (-1)^{n-1}P(1, 1, 1, x, y) \quad \text{by (2,1).}$$

Similarly,

$$Q^2(1, 1, n, x, y) - Q(1, 1, n+1, x, y)Q(1, 1, n-1, x, y) = 4(x^2 + 1)P(1, 1, 1, x, y). \tag{3.11}$$

More complicated algebra, with the use of the above method, produces the *generalized Simson's formula* analogues, namely,

$$P^2(1, 1, n, x, y) - P(1, 1, n+r, x, y)P(1, 1, n-r, x, y) = (-1)^{n-r}P_r^2 P(1, 1, 1, x, y) \tag{3.12}$$

and

$$Q^2(1, 1, n, x, y) - Q(1, 1, n+r, x, y)Q(1, 1, n-r, x, y) = (-1)^{n+r+1}4(x^2 + 1)P_r^2 P(1, 1, 1, x, y). \tag{3.13}$$

Other interesting results may be established by the methods exhibited, for example,

$$P(1, 1, 2n, x, y) = \frac{1}{2}\{P_n Q(1, 1, n, x, y) + Q_n P(1, 1, n, x, y)\}. \tag{3.14}$$

The above information represents a small sample of knowledge available to us. However, the algebra becomes quite awkward when the more general generating functions (2.1) and (2.2) are exploited in that context.

4. SUMS OF GENERATING FUNCTIONS

Let us now consider series whose terms are generating functions.

Summing in (3.1) used as a difference equation and tidying up, we come to

$$\sum_{r=1}^{n} P(1, 1, r, x, y) = \{P(1, 1, n+1, x, y) + P(1, 1, n, x, y) - P(1, 1, 1, x, y) - P(1, 1, 0, x, y)\}/2x. \tag{4.1}$$

For variation, consider next a matrix approach. Accordingly, by (3.6) applied repeatedly,

$$\begin{aligned}\sum_{r=1}^{n} Q(1, 1, r, x, y) &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.2)\\ &= [1 \quad 0][I + P + P^2 + \cdots + P^{n-1}]\begin{bmatrix} Q(1, 1, 1, x, y) \\ Q(1, 1, 0, x, y)\end{bmatrix} \\ &= \frac{1}{2x}[1 \quad 0]\begin{bmatrix} P_{n+1} + P_n - 1 & P_n + P_{n-1} - 1 \\ P_n + P_{n-1} - 1 & P_{n-1} + P_{n-2} - 2x - 1\end{bmatrix}\begin{bmatrix} Q(1, 1, 1, x, y) \\ Q(1, 1, 0, x, y)\end{bmatrix} \\ &= \{Q(1, 1, n+1, x, y) + Q(1, 1, n, x, y) - Q(1, 1, 1, x, y) - Q(1, 1, 0, x, y)\}/2x,\end{aligned}$$

by (3.7), (1.19), and [6, equation (2.11)].

Parallel treatments produce

$$\begin{aligned}\sum_{r=1}^{n} (-1)^r P(1, 1, r, x, y) &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.3)\\ &= \{(-1)^n P(1, 1, n+1, x, y) + (-1)^{n-1} P(1, 1, n, x, y) - P(1, 1, 1, x, y) + P(1, 1, 0, x, y)\}/2x\end{aligned}$$

and

$$\begin{aligned}\sum_{r=1}^{n} (-1)^r Q(1, 1, r, x, y) &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.4)\\ &= \{(-1)^n Q(1, 1, n+1, x, y) + (-1)^{n-1} Q(1, 1, n, x, y) - Q(1, 1, 1, x, y) + Q(1, 1, 0, x, y)\}/2x.\end{aligned}$$

Extensions of the above theory may be exhibited (see [7]) for

$$P(1, m, mr+k, x, y, z) = \sum_{r=0}^{\infty} P(1, m, mr+k, x, y)z^r \tag{4.5}$$

with a similar formulation for the Pell-Lucas generating functions.

5. GENERATING FUNCTIONS FOR SECOND POWERS OF PELL POLYNOMIALS

Exploiting (1.16) as a difference equation, we may demonstrate that, ultimately,

$$(1 - Q_2 y + y^2)\sum_{r=0}^{\infty} P_r^2 y^r \tag{5.1}$$

$$= -y + 2y - 2y^2 + 2y^3 - \cdots + 2(-1)^{r-1}y^r + \cdots$$

$$= \frac{y - y^2}{1 + y},$$

whence

$$\sum_{r=0}^{\infty} P_r y^r = \frac{y - y^2}{1 - (4x^2 + 1)y - (4x^2 + 1)y^2 + y^3}, \tag{5.2}$$

that is,

$$P(2,\ 1,\ 0,\ x,\ y) = (y - y^2)\Delta^{(2)} \quad \text{by (1.12)}. \tag{5.2$'$}$$

Similarly,

$$Q(2,\ 1,\ 0,\ x,\ y) = (4 - (12x^2 + 4)y - 4x^2y^2)\Delta^{(2)}. \tag{5.3}$$

One may also show that

$$\sum_{r=0}^{\infty} P_{r+1}P_{r+2}y^r = 2x\Delta^{(2)} \tag{5.4}$$

and

$$\sum_{r=0}^{\infty} Q_{r+1}Q_{r+2}y^r = 2x\{(4x^2 + 2) + 2(4x^2 + 2)y - 2y^2\}\Delta^{(2)}. \tag{5.5}$$

Generalizations of (5.2) and (5.3) to expressions for $P(2, 1, m, x, y)$ and $Q(2, 1, m, x, y)$ are obtainable (see [7]). In particular,

$$P(2,\ 1,\ 1,\ x,\ y) = (1 - y)\Delta^{(2)}, \tag{5.6}$$

while

$$Q(2,\ 1,\ 2,\ x,\ y) = \{(4x^2 + 2)^2 + (16x^4 + 4x^2 - 4)y - 4x^2y^2\}\Delta^{(2)}. \tag{5.7}$$

Note in passing the marginally useful result that

$$P(2,\ 1,\ 1,\ x,\ y) - P(2,\ 1,\ 0,\ x,\ y) = (1 - y)^2\Delta^{(2)}, \tag{5.8}$$

which has an application in some complicated algebra elsewhere [7].

The theory outlined above extends (though not easily) to $P(2, 1, m, x, y)$ [and $Q(2, 1, m, x, y)$], and more generally to $P(2, m, mr + k, x, y)$. A difference equation resulting from this algebraic maelstrom, and which is useful in deriving fresh information, is

$$P(2,\ m,\ m + k,\ x,\ y) - Q_{2m}P(2,\ m,\ k,\ x,\ y) + P(2,\ m,\ -m + k,\ x,\ y) \tag{5.9}$$

$$= \frac{2(-1)^k P_m^2}{1 + y}.$$

6. GENERATING FUNCTIONS FOR CUBES OF PELL POLYNOMIALS

With care, we may demonstrate the validity of

$$P_{n+1}^3 - Q_3 P_n^3 - P_{n-1}^3 = (-1)^n 6xP_n. \tag{6.1}$$

Use this for summing to derive, first [cf. (1.9) and (1.12)],

$$(1 - Q_3 y - y^2)\sum_{r=0}^{\infty} P_r^3 y^r = y - 6xy^2\Delta', \tag{6.2}$$

whence

$$P(3, 1, 0, x, y) = (y - 4xy^2 - y^3)\Delta^{(3)} \tag{6.3}$$

in which

$$\Delta^{(3)}(1 - Q_3 y - y^2) = \Delta'. \tag{6.4}$$

Similarly,

$$Q(3, 1, 0, x, y) = \{8 - (56x^3 + 32x)y - (64x^4 + 48x^2 + 8)y^2 + 8x^3y^3\}\Delta^{(3)}. \tag{6.5}$$

Indulging in an orgy of algebra, we may construct (see [7]) a generalization of (6.1) relating to P_{mr+k}^3 as leading term. Ultimately, we establish a formula for $P(3, m, k, x, y)$, the generating function for P_{mr+k}^3, although it it not a pretty sight.

For possible interest we append the expression for $\Delta_{(m)}^{(3)}$, namely, cf. (1.13) also,

$$\Delta_{(m)}^{(3)} = \begin{bmatrix} 1 - \{Q_{3m} + (-1)^m Q_m\}y + (-1)^m\{Q_m Q_{3m} + 2\}y^2 \\ -(-1)^m\{Q_{3m} + (-1)^m Q_m\}y^3 + y^4 \end{bmatrix}^{-1} \tag{6.6}$$

Obviously, the foregoing theory could be developed almost *ad infinitum ad nauseam* for $P(j, m, k, x, y)$. Patience, skill, and motivation would be required for this task.

7. GENERATING FUNCTIONS FOR DIAGONAL FUNCTIONS

Rising diagonal functions R_n for $\{P_n\}$ and r_n for $\{Q_n\}$ were defined in [6]. *Descending diagonal functions* D_n and d_n for these polynomials also exist (see [7]). Work on these types of functions, but for other polynomials, may be found in [3], [4], and [5].

Write

$$D \equiv D(x, y) = \sum_{n=1}^{\infty} D_n y^{n-1}, \tag{7.1}$$

$$d \equiv d(x, y) = \sum_{n=2}^{\infty} d_n y^{n-1}, \tag{7.2}$$

$$R \equiv R(x, y) = \sum_{n=1}^{\infty} R_n y^{n-1}, \tag{7.3}$$

$$r \equiv r(x,\ y) = 1 + \sum_{n=2}^{\infty} r_n y^{n-1}. \tag{7.4}$$

Then, following [3]-[5], we find

$$D = \frac{1}{1 - (2x + 1)y}, \tag{7.5}$$

$$d = \frac{2x + 2}{1 - (2x + 1)y}, \tag{7.6}$$

$$R = \frac{1}{1 - 2xy - y^3}, \tag{7.7}$$

$$r = \frac{1 + y^3}{1 - 2xy - y^3}. \tag{7.8}$$

Furthermore,

$$\sum_{n=1}^{\infty} D_{2n} y^{n-1} = \frac{2x + 1}{1 + (2x + 1)^2 y} \tag{7.9}$$

and

$$\sum_{n=1}^{\infty} D_{2n-1} y^{n-1} = \frac{1}{1 + (2x + 1)^2 y}. \tag{7.10}$$

Partial differentiation yields

$$2y\frac{\partial D}{\partial y} - (2x + 1)\frac{\partial D}{\partial x} = 0, \tag{7.11}$$

$$2y\frac{\partial d}{\partial y} - (2x + 1)\left(\frac{\partial d}{\partial x} - 2D\right) = 0, \tag{7.12}$$

$$2y\frac{\partial R}{\partial y} - (2x + 3y^2)\frac{\partial R}{\partial x} = 0, \tag{7.13}$$

$$2y\frac{\partial r}{\partial y} - (2x + 3y^2)\frac{\partial r}{\partial x} - 6(r - R) = 0. \tag{7.14}$$

8. CONCLUDING REMARKS

Information provided above is merely "the tip of the iceberg." Much more lies to be discovered by effort and enterprise.

Clearly, there exists a corresponding investigation involving exponential generating functions.

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