SOME PROPERTIES OF THE GENERALIZATION OF THE FIBONACCI SEQUENCE

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1. INTRODUCTION

Let the arbitrary real numbers a, b, c, and d be given. Construct two sequences $\{X_n\}$ and $\{Y_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = Y_{n+1} + Y_n \\ Y_{n+2} = X_{n+1} + X_n \end{cases} \quad (n \geqslant 0). \tag{1}$$

Clearly, if we set $a = b$ and $c = d$, then the sequences $\{X_n\}$ and $\{Y_n\}$ will coincide with each other and with the sequence $\{F_n(a, c)\}$.

In 1985, K. T. Atanassov, L. C. Atanassova, & D. D. Sasselov [1] showed that

$$X_{n+2} = \frac{1}{2}\left\{\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)a + \left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right)b\right.$$
$$\left. + \left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right)c + \left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right)d\right\}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$, for $n \geqslant 0$. (2)

2. THE GENERALIZATION OF THE FIBONACCI SEQUENCE

Consider the generalized recursive form of (1), as follows:

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = r_1X_{n+1} + r_2X_n + r_3Y_{n+1} + r_4Y_n \\ Y_{n+2} = r_1Y_{n+1} + r_2Y_n + r_3X_{n+1} + r_4X_n \end{cases} \quad (n \geqslant 0), \tag{3}$$

where r_i is real.

Define

$$\begin{cases} X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d \\ Y_n = Y_{n,1}a + Y_{n,2}b + Y_{n,3}c + Y_{n,4}d \\ U_n = X_n + Y_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d \\ V_n = X_n - Y_n = V_{n,1}a + V_{n,2}b + V_{n,3}c + V_{n,4}d \end{cases}$$

then $\{U_n\}$ and $\{V_n\}$ can be defined by the recursions:

$$\begin{cases} U_0 = a + b,\ U_1 = c + d \text{ and } U_{n+2} = (r_1 + r_3)U_{n+1} + (r_2 + r_4)U_n \\ V_0 = a - b,\ V_1 = c - d \text{ and } V_{n+2} = (r_1 - r_3)V_{n+1} + (r_2 - r_4)V_n \end{cases}$$

i.e.,

$$\begin{cases} U_n = W_n(a + b, c + d; r_1 + r_3, -r_2 - r_4) \\ V_n = W_n(a - b, c - d; r_1 - r_3, r_4 - r_2) \end{cases} \qquad \text{(See [2, 3].)}$$

Since

$$\begin{aligned} U_n(a, b, c, d) &= X_n(a, b, c, d) + Y_n(a, b, c, d) \\ &= X_n(a, b, c, d) + X_n(b, a, d, c), \text{ by symmetrical property,} \\ &= X_n(a + b, a + b, c + d, c + d) \\ &= X_{n,1}(a + b) + X_{n,2}(a + b) + X_{n,3}(c + d) + X_{n,4}(c + d) \\ &= (X_{n,1} + X_{n,2})a + (X_{n,1} + X_{n,2})b + (X_{n,3} + X_{n,4})c \\ &\quad + (X_{n,3} + X_{n,4})d \end{aligned}$$

and

$$\begin{aligned} V_n(a, b, c, d) = (X_{n,1} - X_{n,2})a + (X_{n,2} - X_{n,1})b &+ (X_{n,3} - X_{n,4})c \\ &+ (X_{n,4} - X_{n,3})d, \end{aligned}$$

compare with the coefficients of a, b, c, and d, we obtain:

$$\begin{cases} U_{n,1} = U_{n,2} = W_n(1, 0; r_1 + r_3, -r_2 - r_4) \\ \qquad = \displaystyle\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(r_1 + r_3)^{n-2k}(r_2 + r_4)^k \\ U_{n,3} = U_{n,4} = W_n(0, 1; r_1 + r_3, -r_2 - r_4) \\ \qquad = \displaystyle\sum_{k=1}^{[(n+1)/2]} \binom{n-k}{k-1}(r_1 + r_3)^{n-2k+1}(r_2 + r_4)^{k-1} \\ V_{n,1} = -V_{n,2} = W_n(1, 0; r_1 - r_3, -r_2 + r_4) \\ \qquad = \displaystyle\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(r_1 - r_3)^{n-2k}(r_2 - r_4)^k \\ V_{n,3} = -V_{n,4} = W_n(0, 1; r_1 - r_3, -r_2 + r_4) \\ \qquad = \displaystyle\sum_{k=1}^{[(n+1)/2]} \binom{n-k}{k-1}(r_1 - r_3)^{n-2k+1}(r_2 - r_4)^{k-1} \end{cases} \tag{4}$$

for $n \geqslant 2$.

Hence,

$$\begin{cases} U_n = (a + b)U_{n,1} + (c + d)U_{n,3} \\ V_n = (a - b)V_{n,1} + (c - d)V_{n,3}. \end{cases}$$

Since $U_n = X_n + Y_n$ and $V_n = X_n - Y_n$, thus,

$$\begin{cases} X_n = (U_n + V_n)/2 \\ Y_n = (U_n - V_n)/2 \end{cases}$$

is the solution of (3).

Example 1: Let $r_1 = r_2 = 0$ and $r_3 = r_4 = 1$. Then, we have:

$$\begin{cases} U_{n,1} = W_n(1, 0; 1, -1) = F_{n-1} \\ U_{n,3} = W_n(0, 1; 1, -1) = F_n \\ V_{n,1} = W_n(1, 0; -1, 1) = [1, 0, -1] \\ V_{n,3} = W_n(0, 1; -1, 1) = [0, 1, -1] \end{cases}$$

where $[t_1, t_2, \ldots, t_k] = t_j$ if $n \equiv j \pmod{k}$.

Hence,

$$X_n = \{(F_{n-1} + [1, 0, -1])a + (F_{n-1} + [-1, 0, 1])b + (F_n + [0, 1, -1])c + (F_n + [0, -1, 1])d\}/2 \tag{5}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (1), where F_i is the i^{th} Fibonacci number. Note that (5) is the simple form of (2).

Example 2: Let $r_3 = r_4 = 0$ and $r_1 = r_2 = 1$. Then, we have:

$$U_{n,1} = V_{n,1} = W_n(1, 0; 1, -1) = F_{n-1}$$

and

$$U_{n,3} = -V_{n,3} = W_n(0, 1; 1, -1) = F_n.$$

Thus,

$$\begin{cases} X_n = F_{n-1}a + F_nc \\ Y_n = F_{n-1}b + F_nd \end{cases}$$

is the solution of (3) in [1].

Example 3: Let $r_1 = r_4 = 0$ and $r_2 = r_3 = 1$. Then, we have:

$$U_{n,1} = W_n(1, 0; 1, -1) = (-1)^n V_{n,1} = F_{n-1}$$

and

$$U_{n,3} = W_n(0, 1; 1, -1) = (-1)^n V_{n,3} = F_n.$$

Thus,

$$\begin{aligned} X_n &= \{(1 + (-1)^n)F_{n-1}a + (1 - (-1)^n)F_{n-1}b + (1 - (-1)^n)F_nc \\ &\qquad + (1 + (-1)^n)F_nd\}/2 \\ &= \begin{cases} F_{n-1}a + F_nd, & n \text{ even} \\ F_{n-1}b + F_nc, & n \text{ odd} \end{cases} \end{aligned}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (4) in [1].

Example 4: Let $r_2 = r_3 = 0$ and $r_1 = r_4 = 1$. Then, we have:

$$\begin{aligned} U_{n,1} &= W_n(1, 0; 1, -1) = F_{n-1} \\ U_{n,3} &= W_n(0, 1; 1, -1) = F_n \\ V_{n,1} &= W_n(1, 0; 1, 1) = [1, 0, -1, -1, 0, 1] \\ V_{n,3} &= W_n(0, 1; 1, 1) = [0, 1, 1, 0, -1, -1] \end{aligned}$$

Thus,

$$\begin{aligned} X_n = \{&(F_{n-1} + [1, 0, -1, -1, 0, 1])a + (F_{n-1} + [-1, 0, 1, 1, 0, -1])c \\ &+ (F_n + [0, 1, 1, 0, -1, -1])c + (F_n + [0, -1, -1, 0, 1, 1])d\}/2 \end{aligned}$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of (5) in [1].

Example 5: Let $r_1 = r_2 = 0$, $r_3 = 2$, and $r_4 = 1$. Then, we have:

$$U_{n,1} = W_n(1, 0; 2, -1) = \{(2 - \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n\}/4$$
$$U_{n,3} = W_n(0, 1; 2, -1) = \sqrt{2}\{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\}/4$$
$$V_{n,1} = W_n(1, 0; -2, 1) = (n - 1)(-1)^{n-1}$$
$$V_{n,3} = W_n(0, 1; -2, 1) = n(-1)^{n-1}$$

Thus,

$$X_{n,1} = \{(2 - \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n + 4(n - 1)(-1)^{n-1}\}/8$$
$$X_{n,2} = \{(2 + \sqrt{2})(1 + \sqrt{2})^n + (2 + \sqrt{2})(1 - \sqrt{2})^n + 4(n - 1)(-1)^n\}/8$$
$$X_{n,3} = X_{n+1,2} = \{\sqrt{2}(1 + \sqrt{2})^n - \sqrt{2}(1 - \sqrt{2})^n + 4n(-1)^n\}/8$$
$$X_{n,4} = X_{n+1,1} = \{\sqrt{2}(1 + \sqrt{2})^n - \sqrt{2}(1 - \sqrt{2})^n + 4n(-1)^{n-1}\}/8$$

Hence,

$$X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d$$

and $Y_n(a, b, c, d) = X_n(b, a, d, c)$ is the solution of the following system:

$$\begin{cases} X_0 = a,\ X_1 = c,\ Y_0 = b,\ Y_1 = d, \\ X_{n+2} = 2Y_{n+1} + Y_n \\ Y_{n+2} = 2X_{n+1} + X_n \end{cases} \quad (n \geqslant 0)$$

By the five examples above and (4), we obtain the following formulas:

$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} = F_{n-1}$$
$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(-1)^k = [1, 0, -1, -1, 0, 1] = (-1)^n[1, 0, -1]$$
$$\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1}(-1)^{k-1}2^{n-2k} = n - 1.$$

3. THE TRIBONACCI SEQUENCE

Let the arbitrary real numbers a, b, c, d, e, and h be given. Construct two sequences $\{X_n\}$ and $\{Y_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = b,\ X_2 = c,\ Y_0 = d,\ Y_1 = e,\ Y_2 = h, \\ X_{n+3} = Y_{n+2} + Y_{n+1} + Y_n \\ Y_{n+3} = X_{n+2} + X_{n+1} + X_n \end{cases} \quad (n \geqslant 0). \tag{6}$$

Define:

$$X_n = X_{n,1}a + X_{n,2}b + X_{n,3}c + X_{n,4}d + X_{n,5}e + X_{n,6}h$$
$$Y_n = Y_{n,1}a + Y_{n,2}b + Y_{n,3}c + Y_{n,4}d + Y_{n,5}e + Y_{n,6}h$$
$$U_n = X_n + Y_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d + U_{n,5}e + U_{n,6}h$$
$$V_n = X_n - Y_n = V_{n,1}a + V_{n,2}b + V_{n,3}c + V_{n,4}d + V_{n,5}e + V_{n,6}h$$

Then, we have:

$$U_{n,1} = U_{n,4} \qquad V_{n,1} = -V_{n,4} = [1, 0, 0, -1]$$
$$U_{n,2} = U_{n,5} = U_{n+2,1} - U_{n+1,1} \qquad V_{n,2} = -V_{n,5} = [0, 1, 0, -1]$$
$$U_{n,3} = U_{n,6} = U_{n+1,1} \qquad V_{n,3} = -V_{n,6} = [0, 0, 1, -1]$$

where $\{U_{n,1}\}$ can be defined by the recursions (cf. the definition of the Tribonacci numbers in [4] and [5]):

$$U_{0,1} = 1,\ U_{1,1} = U_{2,1} = 0, \text{ and } U_{n+3,1} = U_{n+2,1} + U_{n+1,1} + U_{n,1},$$

for $n \geqslant 0$. That is to say,

$$U_{0,1} = 1,\ U_{1,1} = U_{2,1} = 0,\ U_{3,1} = 1,$$

and

$$U_{n+4,1} = 2U_{n+3,1} - U_{n,1} \quad (n \geqslant 0),$$

since $x^3 - x^2 - x - 1$ is a factor of $x^4 - 2x^3 + 1$. Thus, we have:

$$X_{n,1} = (U_{n,1} + V_{n,1})/2 = (U_{n,1} + [1, 0, 0, -1])/2$$
$$X_{n,2} = (U_{n,2} + V_{n,2})/2 = (U_{n+2,1} - U_{n+1,1} + [0, 1, 0, -1])/2$$
$$X_{n,3} = (U_{n,3} + V_{n,3})/2 = (U_{n+1,1} + [0, 0, 1, -1])/2$$
$$X_{n,4} = (U_{n,4} + V_{n,4})/2 = (U_{n,1} + [-1, 0, 0, 1])/2$$
$$X_{n,5} = (U_{n,5} + V_{n,5})/2 = (U_{n+2,1} - U_{n+1,1} + [0, -1, 0, 1])/2$$
$$X_{n,6} = (U_{n,6} + V_{n,6})/2 = (U_{n+1,1} + [0, 0, -1, 1])/2$$

Hence,

$$X_n = \{(a+d)U_{n,1} + (c+h-b-d)U_{n+1,1} + (b+d)U_{n+2,1} + [a-d,\ b-e,\ c-h,\ d+e+h-a-b-c]\}/2$$

and $Y_n(a, b, c, d, e, h) = X_n(d, e, h, a, b, c)$ is the solution of (6).

4. THE FIBONACCI-TRIPLES SEQUENCE

Let the arbitrary real numbers a, b, c, d, e, and h be given. Construct three sequences $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ for which

$$\begin{cases} X_0 = a,\ X_1 = b,\ Y_0 = c,\ Y_1 = d,\ Z_0 = e,\ Z_1 = h, \\ X_{n+2} = Y_{n+1} + Z_n \\ Y_{n+2} = Z_{n+1} + X_n \quad (n \geqslant 0). \\ Z_{n+2} = X_{n+1} + Y_n \end{cases} \tag{7}$$

The first ten terms of the sequences defined in (7) are shown below:

n	X_n	Y_n	Z_n	n	X_n	Y_n	Z_n
0	a	c	e				
1	b	d	h	6	$8h + 5a$	$8b + 5c$	$8d + 5e$
2	$d + e$	$h + a$	$b + c$	7	$13b + 8c$	$13d + 8e$	$13h + 8a$
3	$2h + a$	$2b + c$	$2d + e$	8	$21d + 13e$	$21h + 13a$	$21b + 13c$
4	$3b + 2c$	$3d + 2e$	$3h + 2a$	9	$34h + 21a$	$34b + 21c$	$34d + 21e$
5	$5d + 3e$	$5h + 3a$	$5b + 3c$	10	$55b + 34c$	$55d + 34e$	$55h + 34a$

Define

$$U_n = X_n + Y_n + Z_n = U_{n,1}a + U_{n,2}b + U_{n,3}c + U_{n,4}d + U_{n,5}e + U_{n,6}h,$$

then $\{U_n\}$ can be defined by the recursion

$$U_0 = a + c + e,\ U_1 = b + d + h, \text{ and } U_{n+2} = U_{n+1} + U_n,$$

i.e., $U_n = W_n(a + c + e,\ b + d + h;\ 1, -1)$.

Compare with the coefficients of a, b, c, d, e, and h. We have:

$$U_{n,1} = U_{n,3} = U_{n,5} = W_n(1, 0; 1, -1) = F_{n-1}$$

$$U_{n,2} = U_{n,4} = U_{n,6} = W_n(0, 1; 1, -1) = F_n$$

Thus,

$$U_n = (a + c + e)F_{n-1} + (b + d + h)F_n.$$

Since $X_n = Y_{n+2} - Z_{n+1}$ and $X_n = Z_{n+1} - Y_{n-1}$, we have:

$$\begin{cases} X_n = (Y_{n+2} - Y_{n-1})/2 \\ Y_n = (Z_{n+2} - Z_{n-1})/2 \\ Z_n = (X_{n+2} - X_{n-1})/2 \end{cases}$$

Since $X_n = Y_{n-1} + Z_{n-2}$ and $X_n = Z_{n+1} - Y_{n-1}$, we obtain:

$$\begin{cases} X_n = (Z_{n+1} + Z_{n-2})/2 \\ Y_n = (X_{n+1} + X_{n-2})/2 \\ Z_n = (Y_{n+1} + Y_{n-2})/2 \end{cases}$$

Since $4X_{n+3} = 2(Y_{n+5} - Y_{n+2}) = (X_{n+6} + X_{n+3}) - (X_{n+3} + X_n) = X_{n+6} - X_n$, we have:

$$\begin{cases} X_{n+6} = 4X_{n+3} + X_n \\ Y_{n+6} = 4Y_{n+3} + Y_n \\ Z_{n+6} = 4Z_{n+3} + Z_n \end{cases}$$

When $n \equiv 0 \pmod 3$, taking $n = 3m$, we have:

$$X_{3(m+2)} = 4X_{3(m+1)} + X_{3m} \text{ with } X_0 = a \text{ and } X_3 = 2h + a.$$

Letting $V_m = X_{3m}$, we have:

$$V_{m+2} = 4V_{m+1} + V_m \text{ with } V_0 = a \text{ and } V_1 = 2h + a.$$

Therefore, we get:

$$\begin{aligned} V_m &= \frac{2h + (\sqrt{5} - 1)a}{2\sqrt{5}}(2 + \sqrt{5})^m + \frac{(\sqrt{5} + 1)a - 2h}{2\sqrt{5}}(2 - \sqrt{5})^m \\ &= \frac{1}{\sqrt{5}}\left\{\left(\left(\frac{1 + \sqrt{5}}{2}\right)^{3m-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{3m-1}\right)a + \left(\left(\frac{1 - \sqrt{5}}{2}\right)^{3m} - \left(\frac{1 - \sqrt{5}}{2}\right)^{3m}\right)h\right\}, \\ &= F_{3m-1}a + F_{3m}h \quad \text{by } \left(\frac{1 \pm \sqrt{5}}{2}\right)^3 = 2 \pm \sqrt{5}, \end{aligned}$$

i.e., $X_n = F_{n-1}a + F_nh$.

Using a similar method, we have: $X_n = \begin{cases} F_{n-1}a + F_nh, & \text{if } n \equiv 0 \pmod 3 \\ F_{n-1}c + F_nb, & \text{if } n \equiv 1 \pmod 3 \\ F_{n-1}e + F_nd, & \text{if } n \equiv 2 \pmod 3 \end{cases}$

$$Y_n(a, b, c, d, e, h) = X_n(e, h, a, b, c, d)$$

and

$$Z_n(a, b, c, d, e, h) = X_n(c, d, e, h, a, b)$$

as the solution of (7).

Numerous similar pairs of sequences can be constructed. However, the ones introduced here stand most closely to the very spirit of the Tribonacci sequence (or the Fibonacci-triples sequence) and its generalization rules.

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