A CONSTELLATION OF SEQUENCES OF GENERALIZED PELL POLYNOMIALS

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1. INTRODUCTION

In [1] and [2], Byrd introduced a sequence of polynomials which we call Pell. These polynomials may be defined, in the first instance, thus:

$$\begin{cases} p_0(x) = 0,\ p_1(x) = 1, \\ p_{n+1}(x) = 2xp_n(x) + p_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.1}$$

The polynomials cognate to these, the Pell-Lucas, may be defined thus:

$$\begin{cases} q_0(x) = 2,\ q_1(x) = 2x, \\ q_{n+1}(x) = 2xq_n(x) + q_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.2}$$

These two sequences have been studied in more detail in [5]-[10]. The Binet formulas for the two sequences of polynomials are

$$p_n(x) = \frac{\eta^n - \psi^n}{\eta - \psi} \tag{1.3}$$

and

$$q_n(x) = \eta^n + \psi^n \tag{1.4}$$

where η, ψ are roots of the equation

$$y^2 - 2xy - 1 = 0. \tag{1.5}$$

Hence, η, ψ are given by

$$\eta = x + \sqrt{(x^2 + 1)}, \quad \psi = x - \sqrt{(x^2 + 1)}. \tag{1.6}$$

In [12]-[14], Walton, and Walton & Horadam have studied a sequence of generalized Pell polynomials. They are defined thus:

$$\begin{cases} A_0(x) = q,\ A_1(x) = p, \\ A_{n+1}(x) = 2xA_n(x) + A_{n-1}(x), \text{ for } n \geqslant 1. \end{cases} \tag{1.7}$$

Another sequence of generalized Pell polynomials or, rather, a constellation of them is proposed here.

2. FIRST ENCOUNTER WITH THE CONSTELLATION OF SEQUENCES OF GENERALIZED PELL POLYNOMIALS

This constellation was first encountered in an effort to replicate for Pell polynomials what Gould [3] and others had done with a formula of Lucas.

An important identity for $p_n(x)$, easily proved from Binet formulas (1.3) and (1.4) is:

$$p_{n+m}(x) - q_m(x)p_n(x) + (-)^m p_{n-m}(x) = 0 \tag{2.1}$$

This may be regarded as a generalization for (1.1). By repeated applications of (2.1), we get:

$$\begin{cases} p_n(x) \\ = q_m(x)p_{n-m}(x) + (-)^{m-1}p_{n-2m}(x) \\ = (q_m^2(x) + (-)^{m-1})p_{n-2m}(x) + (-)^{m-1}q_m(x)p_{n-3m}(x) \\ = (q_m^3(x) + 2(-)^{m-1}q_m(x))p_{n-3m}(x) + (-)^{m-1}(q_m(x) + (-)^{m-1})p_{n-4m}(x) \\ = (q_m^4(x) + 3(-)^{m-1}q_m^2(x) + (-)^{2(m-1)})p_{n-4m}(x) + \\ \qquad + (-)^{m-1}(q_m^3(x) + 2(-)^{m-1}q_m(x))p_{n-5m}(x) \end{cases} \tag{2.2}$$

We may present these lines thus:

$$\begin{cases} p_n(x) \\ = p_{1,m}(x)p_n(x) + (-)^{m-1}p_{0,m}(x)p_{n-m}(x) \\ = p_{2,m}(x)p_{n-m}(x) + (-)^{m-1}p_{1,m}(x)p_{n-2m}(x) \\ = p_{3,m}(x)p_{n-2m}(x) + (-)^{m-1}p_{2,m}(x)p_{n-3m}(x) \\ = p_{4,m}(x)p_{n-3m}(x) + (-)^{m-1}p_{3,m}(x)p_{n-4m}(x) \\ = p_{5,m}(x)p_{n-4m}(x) + (-)^{m-1}p_{4,m}(x)p_{n-5m}(x) \end{cases} \tag{2.3}$$

where

$$\begin{cases} p_{0,m}(x) = 0 \\ p_{1,m}(x) = 1 \\ p_{2,m}(x) = q_m(x) \\ p_{3,m}(x) = q_m^2(x) + (-)^{m-1} \\ p_{4,m}(x) = q_m^3(x) + 2(-)^{m-1}q_m(x) \\ p_{5,m}(x) = q_m^4(x) + 3(-)^{m-1}q_m(x) + (-)^{2(m-1)} \end{cases} \tag{2.4}$$

The procedure followed in (2.2) and (2.3) may be continued indefinitely, when allowance is made for the first subscript to be negative. It is clear from (2.2) that

$$p_{n,m}(x) = q_m(x)p_{n-1,m}(x) + (-)^{m-1}p_{n-2,m}(x). \tag{2.5}$$

Starting again, we may define the sequence $\{p_{n,m}(x)\}$ thus:

$$\begin{cases} p_{0,m}(x) = 0,\ p_{1,m}(x) = 1, \\ p_{n+1,m}(x) = q_m(x)p_{n,m}(x) + (-)^{m-1}p_{n-1,m}(x), \text{ for } n \geqslant 1. \end{cases} \tag{2.6}$$

The defining equation gives rise to a constellation of sequences, one for each value of m.

3. SOME IDENTITIES AND GENERATORS FOR THE SEQUENCE $\{p_{n,m}(x)\}$

The results in (2.4) may be used as the basis for a proof by induction of an explicit formula for $p_{n,m}(x)$. It is:

$$p_{n,m}(x) = \sum_{i=0}^{[(n-1)/2]} (-)^{i(m-1)}\binom{n-1-i}{i} q_m^{n-1-2i}(x) \tag{3.1}$$

From this we may show that:

$$q_m^n(x) = \sum_{r=0}^{[n/2]} (-)^{rm} \binom{n}{r} \frac{n - 2r + 1}{n - r + 1} p_{n+1-2r,\,m}(x) \tag{3.2}$$

The Binet formula, also proved by induction, is:

$$p_{n,m}(x) = \frac{\eta^{nm} - \psi^{nm}}{\eta^m - \psi^m} \tag{3.3}$$

where η and ψ are as introduced in (1.6). If the Binet formula were used to define the sequence, negative integral values for n and m are easily introduced.

From (1.3) and (3.3), we have:

$$p_{nm}(x) = p_{n,m}(x)p_m(x) \tag{3.3'}$$

A determinantal generator for $p_{n,m}(x)$ is $\delta_{n,m}(x)$. The determinant is of order n and is defined thus:

$$\delta_{n,m}(x): \begin{cases} d_{rr} = q_m(x) & \text{for } r = 1, 2, \ldots, n \\ d_{r,r+1} = (-)^m & \text{for } r = 1, 2, \ldots, n - 1 \\ d_{r,r-1} = 1 & \text{for } r = 2, 3, \ldots, n \\ d_{rc} = 0 & \text{otherwise} \end{cases} \tag{3.4}$$

where d_{rc} is the entry in the r^{th} row and c^{th} column of $\delta_{n,m}(x)$. One may prove by induction that

$$\delta_{n,m}(x) = p_{n+1,m}(x) \text{ for } n \geqslant 1. \tag{3.5}$$

A matrix generator for $p_{n,m}(x)$ is:

$$\mathscr{P}_m = \begin{bmatrix} q_m(x) & (-)^{m-1} \\ 1 & 0 \end{bmatrix} \tag{3.6}$$

We can easily show, by induction again, that:

$$\mathscr{P}_m^n = \begin{bmatrix} p_{n+1,m}(x) & (-)^{m-1}p_{n,m}(x) \\ p_{n,m}(x) & (-)^{m-1}p_{n-1,m}(x) \end{bmatrix} \tag{3.7}$$

The matrix $\mathscr{P}_m$ has been employed to establish several identities. There are other matrix generators for the sequence.

An algebraic generator is

$$\sum_{n=0}^{\infty} p_{n+1,m}(x) = 1/(1 - q_m(x)y + (-)^m y^2), \tag{3.8}$$

and an exponential generator is:

$$\sum_{n=0}^{\infty} p_{n,m}(x)y^n/n! = \frac{e^{\eta^m y} - e^{\psi^m y}}{\eta^m - \psi^m}$$

The justification for regarding $\{p_{n,m}(x)\}$ as a generalization for $\{p_n(x)\}$ is that, when we put $m = 1$ in the results given above and in others, we obtain the corresponding formulas for the Pell polynomials. First and foremost, we have

$$p_{n,1}(x) = p_n(x). \tag{3.10}$$

We mention, finally, in this section two identities which have been proved by using the matrix $\mathscr{P}_m$. They are the Simson formula and its generalization for $p_{n,m}(x)$.

$$p_{n+1,m}(x)p_{n-1,m}(x) - p^2_{n,m}(x) = (-)^{m(n-1)+1} \tag{3.11}$$

$$p_{n+r,m}(x)p_{n-r,m}(x) - p^2_{n,m}(x) = (-)^{m(n-r)+1}p^2_{r,m}(x) \tag{3.12}$$

4. RELATIONS OF $\{p_{n,m}(x)\}$ WITH CHEBYSHEV POLYNOMIALS

In [1], [2], [5], [6], and [7] some relations of Pell and Pell-Lucas polynomials with Chebyshev polynomials were explored. If we regard $\{p_{n,m}(x)\}$ as a generalization of Pell polynomials, then we would also expect that it should have connections. However, we need to construct first a generalization for Chebyshev polynomials of the second kind [11]. These are $\{U_{n,m}(x)\}$ defined in the following manner:

$$\begin{aligned} &U_{0,m}(x) = 1,\ U_{1,m}(x) = 2T_m(x),\\ &U_{n+1,m}(x) = 2T_m(x)U_{n,m}(x) - U_{n-1,m}(x),\ \text{for } n \geqslant 1, \end{aligned} \tag{4.1}$$

where $T_m(x)$ is the m^{th} Chebyshev polynomial of the first kind [11].

With this definition, it is possible to prove by induction that

$$U_{n,m}(x) = \sum_{j=0}^{[n/2]} (-)^j \binom{n-j}{j} (2T_m(x))^{n-2j},\ \text{for } n \geqslant 1. \tag{4.2}$$

Following from (4.2), we can prove that

$$p_{n,m}(x) = (-i)^{(n-1)m}U_{n-1,m}(ix). \tag{4.3}$$

A hypergeometric representation for $p_{n,m}(x)$ follows from (4.3). It is

$$p_{n,m}(x) = n\,{}_2F_1(n+1, -n+1; 3/2; Y_m)/i^{(n-1)m} \tag{4.4}$$

where

$$Y_m = (2 - i^m q_m(x))/4. \tag{4.5}$$

Another explicit expression for $p_{n,m}(x)$ may also be derived from (4.3), namely,

$$p_{n,m}(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} (q_m(x)/2)^{n-1-2k}(X_m/4)^k \tag{4.6}$$

where X_m is the discriminant of the auxiliary equation of $p_{n,m}(x)$, i.e.,

$$y^2 - q_m(x)y + (-)^m = 0. \tag{4.7}$$

This means that

$$X_m = q^2_m(x) + 4(-)^{m-1}. \tag{4.8}$$

Starting from (2.5) and the identity below, easily established from Binet formulas,

$$q_{(n+1)m}(x) - (q^2_m(x) + 4(-)^{m-1})p_{n,m}(x) + (-)^{m-1}q_{(n-1)m}(x) = 0, \tag{4.9}$$

we obtain other explicit expressions for $p_{n,m}(x)$. They are:

$$p_{2n+1,m}(x) = \sum_{k=0}^{n} (-)^{km} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} X_m^{n-k}; \tag{4.10}$$

and

$$p_{2n,m}(x) = \left\{ \sum_{k=0}^{n-1} (-)^{km} \binom{2n-1-k}{k} X_m^{n-1-k} \right\} q_m(x). \tag{4.11}$$

These interesting and aesthetically appealing formulas for the constellation of sequences $\{p_{n,m}(x)\}$ are a sample of the large number that have been obtained.

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