ON ASSOCIATED AND GENERALIZED LAH NUMBERS AND APPLICATIONS TO DISCRETE DISTRIBUTIONS

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1. INTRODUCTION

First, we consider some definitions and preliminary results needed in this study. Ahuja & Enneking [1] have defined the associated Lah numbers $B(n, r, k)$ by

$$B(n, r, k) = (n!/k!)\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\binom{n+ri-1}{n}, \tag{1}$$

where

$B(n, r, k) = 0$ for $k > n$, $B(n, r, 0) = 0$,
$B(n, r, 1) = r(r+1)\ldots(r+n-1)$, $B(n, r, n) = r^n$
and $B(n, 1, k) = |L(n, k)|$,

the signless Lah numbers (see Riordan [12], p. 44).

Ahuja & Enneking have also obtained (see [2]) the following relations for the $B(n, r, k)$'s:

$$B(n+1, r, k) = (n+rk)B(n, r, k) + rB(n, r, k-1), \tag{2}$$

and

$$[B(n, r, k)]^2 > B(n, r, k+1)B(n, r, k-1) \text{ for } k = 2, 3, \ldots, n-1. \tag{3}$$

We now introduce two other equivalent definitions of $B(n, r, k)$. First, we write

$$B(n, r, k) = [(E^r - I)^k y^{[n]}]_{y=0}/k! \quad (k = 1, \ldots, n), \tag{4}$$

where $Ef(x) = f(x+1)$ and I is the unit operator.

Second, we have

$$B(n, r, k) = (n!/k!)\sum_k \prod_{i=1}^{k}\binom{n_i+r-1}{n_i}, \tag{5}$$

where Σ_k denotes the sum over all positive integral values of the n_i's such that $n_1 + \cdots + n_k = n$ and $n = k, k+1, \ldots$.

Equation (5) follows from the following combinatorial identity:

$$\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\binom{n+ri-1}{n} = \sum_k \prod_{i=1}^{k}\binom{n_i+r-1}{n_i}, \tag{6}$$

where the summation in the right-hand member is extended over integral values of each $n_i \geqslant 1$ such that $n_1 + \cdots + n_k = n$ and $n = k, k+1, \ldots$.

Further, let $R(n, r, k)$ be a sequence of real numbers defined by

$$R(n, r, k) = B(n+1, r, k)/B(n, r, k),\ k = 1, 2, \ldots, n, \tag{7}$$

for given n. These numbers are useful in calculating probability functions independent of rapidly growing associated Lah numbers.

Ahuja & Enneking [1] have introduced the generalized Lah numbers $L_{c,r}(n, k)$ defined by:

$$L_{c,r}(n,\ k) = (n!/k!) \sum (-1)^{k-r_1} \frac{k!}{r_1!r_2!\ldots r_{c+2}!} \times \prod_{j=0}^{c} \left[\binom{j+r-1}{j}\right]^{r_{j+2}} \binom{n - \sum_{j=0}^{c} jr_{j+2} + rr_1 - 1}{rr_1 - 1} \tag{8}$$

for integral $c \geqslant 0$, and $n = k(c+1),\ k(c+1)+1,\ \ldots$, where the summation extends over all $r_j > 0$ such that $\sum_{j=1}^{c+2} r_j = k$.

Using the combinatorial identity

$$\sum_J (-1)^{k-r_1} \frac{k!}{r_1!r_2!\ldots r_{c+2}!} \prod_{j=0}^{c} \left[\binom{j+r-1}{j}\right]^{r_{j+2}} \binom{n - \sum_{j=0}^{c} jr_{j+2} + rr_1 - 1}{rr_1 - 1}$$
$$= \sum_K \prod_{i=1}^{k} \binom{x_i + r - 1}{x_i}, \tag{9}$$

for $c > 0$, and $n = k(c+1),\ k(c+1)+1, \ldots$, where $\sum_J$ extends over all $r_j > 0$ such that $\sum_{j=1}^{c+2} r_j = k$ and $\sum_k$ extends over all $x_i > c$ such that $\sum_{i=1}^{k} x_i = n$, we find an alternative representation of the generalized Lah number as

$$L_{c,r}(n,\ k) = (n!/k!) \sum_K \prod_{i=1}^{k} \binom{x_i + r - 1}{x_i}, \tag{10}$$

where $\sum_K$ is extended over all ordered k-tuples $(x_1,\ x_2,\ \ldots,\ x_k)$ of integers $x_i > c$, $i = 1,\ 2,\ \ldots,\ k$ with $x_1 + x_2 + \cdots + x_k = n$.

Section 2 is devoted to the study of properties of associated Lah numbers. Section 3 is concerned with the properties of ratios of associated Lah numbers. Section 4 deals with a discrete probability distribution involving associated Lah numbers via a generalized occupancy problem. Section 5 contains the problem of estimating a parameter of the population discussed in the preceding section. Section 6 discusses limiting forms of the discrete distribution studied in Section 4. Section 7 introduces an inverse probability distribution involving associated Lah numbers. Section 8 considers the definitions and properties of a conditional multivariate distribution involving associated Lah numbers. The last two sections deal with some applications of generalized Lah numbers.

2. SOME PROPERTIES OF $B(n,\ r,\ k)$

We now investigate properties of $B(n,\ r,\ k)$ and their limiting forms.

Property 1:

$$(rx)^{[n]} = \sum_{k=1}^{\infty} B(n,\ r,\ k)(x)_k, \tag{11}$$

where $(rx)^{[n]} = rx(rx+1)\ldots(rx+n-1)$ and $(x)_k = x(x-1)\ldots(x-k+1)$, x being any real number and r a positive integer.

Proof: $(rx)^{[n]} = [E^{rx}y^{[n]}]_{y=0} = [\{I + (E^r - I)\}^x y^{[n]}]_{y=0}$

$$= \sum_{k=0}^{\infty} \binom{x}{k} [(E^r - I)^k y^{[n]}]_{y=0} = \sum_{k=1}^{\infty} B(n,\ r,\ k)(x)_k \text{ from (4).}$$

However, if x is a positive integer, then

$$(rx)^{[n]} = \sum_{k=1}^{\min(x,n)} B(n, r, k)(x)_k. \tag{12}$$

Property 2:

$$1/(x-1)_k = \sum_{n=k}^{\infty} B(n, r, k)/(rx+1)^{[n]}. \tag{13}$$

This can be proved by induction on k.

Property 3:

$$B(n, r, k) = (1/k)\sum_{x=1}^{n-k+1} (n)_x \binom{x+r-1}{x} B(n-x, r, k-1). \tag{14}$$

Property 4:

$$\lim_{r\to 0} B(n, r, k)/r^k = |s(n, k)|, \tag{15}$$

where $|s(n, k)|$ is the signless Stirling number of the first kind.

Property 5:

$$\lim_{r\to\infty} B(n, r, k)/r^n = S(n, k), \tag{16}$$

where $S(n, k)$ denotes the Stirling number of the second kind.

3. SOME PROPERTIES OF $R(n, r, k)$

In this section we study the following properties of $R(n, r, k)$.

Property 1: The sequence (7) satisfies the recurrence relation

$$R(n, r, k) - (n + rk)$$
$$= R(n-1, r, k-1) - [(n + rk - 1)/R(n-1, r, k)] \tag{17}$$

for $1 < k < n$ and for all n, where

$$R(n, r, 1) = (n + r) \quad \text{and} \quad R(n, r, n) = [n(n+1)(r+1)]/2.$$

The relation (17) follows directly from (2).

Property 2: The sequence (7) increases with k for given n and satisfies the inequality

$$R(n, r, k+1) > R(n, r, k), \text{ for } k = 2, 3, \ldots, n-1. \tag{18}$$

This follows immediately from (3).

Property 3: The sequence (7) satisfies the inequality

$$R(n-1, r, k) + 1 \geqslant R(n, r, k) \quad (n = k+1, k+2, \ldots) \tag{19}$$

with equality only for $k = 1$.

Relation (19) is observed from (17). It shows that the ratio $R(n, r, k)$ grows very slowly with n.

4. A DISCRETE PROBABILITY DISTRIBUTION INVOLVING ASSOCIATED LAH NUMBERS

This section is devoted to the study of a discrete probability distribution involving the associated Lah numbers derived via the following generalized occupancy problem.

Suppose n indistinguishable balls are distributed in $r\theta$ cells constituting θ groups of r cells each. Then the probability that k groups are occupied with n_1 balls in one group, n_2 balls in the second group, ..., n_k balls in the k^{th} group, and the remaining $(\theta - k)$ groups are empty is

$$Pr\{K = k \cap N_1 = n_1, \ldots, N_{k-1} = n_{k-1} | n, r, \theta\}$$
$$= n!(\theta)_k \prod_{i=1}^{k} \binom{n_i + r - 1}{n_i} \Big/ \{(r\theta)^{[n]} k!\}, \tag{20}$$

where $(r\theta)^{[n]} = (r\theta)(r\theta + 1) \ldots (r\theta + n - 1)$ and $n_k = n - n_1 - \cdots - n_{k-1}$.

From (20), the probability that k different groups are occupied out of θ groups (without regard to frequencies) is

$$Pr\{K = k|n, r, \theta\} = f_K(k|n, r, \theta)$$
$$= [(\theta)_k/(r\theta)^{[n]}](n!/k!) \sum \prod_{i=1}^{k} \binom{n_i + r - 1}{n_i}, \tag{21}$$

where the summation extends over all positive integral values of $n_1, \ldots, n_{k-1}$ subject to $n > n_1 + \cdots + n_{k-1}$.

Now, using the definition of associated Lah numbers in (5), the probability function (pf) of the random variable K is

$$f_K(k|n, r, \theta) = B(n, r, k)(\theta)_k/(r\theta)^{[n]}, \quad k = 1, \ldots, n. \tag{22}$$

From (11), it follows that

$$\sum_{k=1}^{n} f_K(k|n, r, \theta) = 1,$$

which verifies that $f_K(k|n, r, \theta)$ is a proper pf.

In particular, if $r = 1$ in (22),

$$f_K(k|n, \theta) = |L(n, k)|(\theta)_k/\theta^{[n]}, \quad (k = 1, \ldots, n), \tag{23}$$

where the $|L(n, k)|$'s are the signless Lah numbers.

The probability model (23) describes the distribution of K, the number of occupied cells, when n indistinguishable balls are assigned to θ cells. Analogously, it gives the distribution of K, the number of occupied energy levels, if n like particles (e.g., protons, nuclei, or atoms containing an even number of elementary particles for the Bose-Einstein system of physical statistics) are assigned to θ energy levels.

The pf (22) satisfies the recurrence relation

$$f_K(k|n, r, \theta) = r(\theta - k + 1)f_K(k - 1|n, r, \theta)/[R(n, r, k) - (n + rk)] \tag{24}$$

for $k = 2, 3, \ldots, n$, where $f_K(1|n, r, \theta) = \theta r^{[n]}/(r\theta)^{[n]}$.

Relation (17) seems to be quite useful in preparing a table for $R(n, r, k)$. The values of $R(n, r, k)$ are necessary in computing the pf from (24).

The mean and variance of K are given by:

$$E(K) = \theta[(r\theta)^{[n]} - (r\theta - r)^{[n]}]/(r\theta)^{[n]}; \tag{25}$$

$$E(K(K - 1)) = (\theta)_2[(r\theta)^{[n]} - 2(r\theta - r)^{[n]} + (r\theta - 2r)^{[n]}]/(r\theta)^{[n]}; \tag{26}$$

$$\text{Var}(K) = E(K(K - 1)) + E(K) - [E(K)]^2. \tag{27}$$

5. ESTIMATION OF THE PARAMETER θ OF THE PROBABILITY DISTRIBUTION OF THE PREVIOUS SECTION

Suppose we have a population of θr cells consisting of θ groups of r cells each, in which r is known but θ is unknown. Suppose n indistinguishable balls are randomly distributed in these cells and k groups are found to be occupied. Here K, the number of occupied groups, has probability function (22). We wish to estimate the underlying parameter θ based upon the observed k.

First, following the arguments of Patil [10], we shall show that a uniformly minimum variance unbiased (UMVU) estimator of θ based on the complete sufficient statistic K does not exist. Second, we shall show that, in some special case, a suitable estimator of θ is obtainable. Suppose we proceed heuristically to construct an unbiased estimator $t(K|n, r)$ of θ based on K. Then the condition of unbiasedness

$$E[t(K|n, r)] = \theta \tag{28}$$

yields

$$t(k|n, r) = [R(n, r, k) - n]/r \quad (k = 1, \ldots, n - 1) \tag{29}$$

and

$$B(n, r, n) = 0. \tag{30}$$

But, by definition, $B(n, r, n) = r^n$, and we arrive at a contradiction. Hence, there is no unbiased estimator of θ.

Here the relative bias of $t(K|n, r)$ satisfies

$$E[t(K|n, r)/\theta] - 1 = -[r^{n+1}(\theta)_{n+1}/\{(r\theta)(r\theta)^{[n]}\}]. \tag{31}$$

We observe that

$$[r^{n+1}(\theta)_{n+1}/\{(r\theta)(r\theta)^{[n]}\}] < 1,$$

thus the relative bias approaches zero for moderately large value of n. Further, in practice, the probability of the maximum outcome may be negligibly small. So the use of (29) may often be justified in a special case where the bias of the estimator is not serious, and in such a case the estimate (29) of the parameter θ is obtainable from the recurrence relation

$$\begin{aligned} &t(k|n, r) - k \\ &= [t(k|n - 1, r) - k][rt(k - 1|n - 1, r) + n - 1]/[rt(k|n - 1, r) + n - 1] \end{aligned} \tag{32}$$

where $1 < k < n$ with

$$t(1|n, r) = n + r + 1 \quad \text{and} \quad t(n|n, r) = [n(n - 1)(r + 1)/2r] + n.$$

The above relation follows from (17).

6. TWO LIMITING DISTRIBUTIONS

We now consider two limiting forms of the distribution (22) which are of much practical use.

First, if $r\theta = \phi$ is constant and $r \to 0$ in (22), then $f_K(k|n, r, \theta)$ becomes the limiting distribution

$$f_K(k|n, \phi) = |s(n, k)|\phi^k/\phi^{[n]} \quad (k = 1, \ldots, n), \tag{33}$$

which has application in genetic studies (see Johnson & Kotz, [8], p. 246) and the distribution of the number of hearers directly from a source (see Bartholomew, [4], p. 317). We observe that (33) is a special case of the power series distribution (see [8], p. 85). When $\phi = 1$, (33) reduces to

$$f_K(k|n) = |s(n, k)|/n! \quad (k = 1, \ldots, n) \tag{34}$$

which has been used by Barlow *et al.* ([3], p. 143) in connection with some problems of testing statistical hypotheses under order restrictions. Equation (34) gives the probability that a permutation of n elements picked at random has k cycles.

Second, if $r \to \infty$ in (22), we find

$$f_K(k|n, \theta) = S(n, k)(\theta)_k/\theta^n \quad (k = 1, \ldots, n). \tag{35}$$

This is known as Steven-Craig's distribution (see Patil & Joshi, [11], p. 56) and sometimes called Arfwedson's distribution (see Johnson & Kotz, [7], p. 251). It is a particular case of the factorial series distribution introduced by Berg [5]. It is also useful in the study of the ecology of plants and animals (see Lewontin & Prout, [9] and Watterson, [13]) and in some problems of sample surveys (see Des Raj & Khamis, [6]). In addition, it can be applied to finding the critical values of the empty cell test (see, e.g., Wilks, [14], pp. 433-37].

7. A PROBABILITY MODEL UNDER AN INVERSE SAMPLING SCHEME

We introduce a probability model involving associated Lah numbers under an inverse sampling scheme.

Suppose that, instead of n being fixed and k variable, random distribution of like balls, one at a time, is continued until a predetermined number k, say, of groups have been occupied. Let the required size be n. Then we have a probability model under the inverse sampling scheme having the pf

$$\begin{aligned} h_N(n|k, r, \theta) &= Pr\{N = n|k, r, \theta\} \\ &= rB(n-1, r, k-1)(\theta)_k/(r\theta)^{[n]}, \quad n = k, k+1, \ldots . \end{aligned} \tag{36}$$

It is seen from (13) that

$$\sum_{n=k}^{\infty} h_N(n|k, r, \theta) = 1.$$

The pf (36) is recognized as a special case of inverse factorial series distribution (see [8], p. 88). It satisfies the following recurrence relation:

$$h_N(n|k, r, \theta) = [R(n-2, r, k-1)/(r\theta + n - 1)]h_N(n-1|k, r, \theta), \tag{37}$$

where the $R(n, r, k)$ satisfy (17).

The mean and variance of N are obtained as follows:

$$E(N) = -(r\theta - 1)(\theta)_k \underset{1/r}{\Delta}[1/(\theta - 1/r)_k] \tag{38}$$

and

$$E(N(N+1)) = (r\theta - 1)(r\theta - 2)(\theta)_k \underset{1/r}{\Delta^2}[1/(\theta - 2/r)_k], \tag{39}$$

where $\underset{1/r}{\Delta} f(\theta) = f(\theta + 1/r) - f(\theta)$.

From (38) and (39), $\mathrm{Var}(N)$ can be obtained easily.

Here we note that N is a complete, sufficient statistic for θ. Making use of this statistic, we now consider the problem of estimation.

Arguing as in Section 5, we can show that the UMVU estimator of θ based on N does not exist. However, if we assume $g(N)$ to be an unbiased estimator of θ, then we find that the relative bias of $g(N)$ is:

$$E[g(N)/\theta] - 1 = r^{k-1}(\theta - 1)_{k-2}/(r\theta)^{[k-1]}. \tag{40}$$

This relative bias does not depend upon n. Thus, it cannot be reduced by taking a large sample. Therefore, it is not possible to provide any usable estimate of θ.

8. A CONDITIONAL MULTIVARIATE DISTRIBUTION INVOLVING ASSOCIATED LAH NUMBERS

We now investigate the properties of a conditional multivariate distribution whose pf can be obtained readily from the associated Lah numbers.

From (5), the joint distribution of $\overline{N} = (N_1, \ldots, N_k)$ (given $N_1 + \cdots + N_k + N_{k+1} = n$) is:

$$Pr\left\{\overline{N} = \overline{n} \,\middle|\, \text{each } n_i > 0,\ i = 1, \ldots, k,\ n > \sum_{i=1}^{k} n_i,\ k \text{ and } r \text{ are positive integers}\right\}$$

$$= (n!/(k+1)!)\prod_{i=1}^{k+1}\binom{n_i + r - 1}{n_i}\Big/B(n, r, k+1), \tag{41}$$

where the mass points (the sample points) of $\overline{n}$ are defined by the set:

$$\left\{\overline{n} \,\middle|\, \text{each } n_i > 0,\ n > \sum_{i=1}^{k} n_i,\ k \text{ and } r \text{ are fixed positive integers}\right\}.$$

It represents the pf of $\overline{N}$ (the group frequencies), if $n > r(k+1)$ indistinguishable balls are put into $r(k+1)$ cells constituting $k+1$ groups of r cells each with no group empty.

To find the mean and variance of N_i, we put, for convenience,

$$A(n, r, k+1) = [r/B(n, r, k+1)]\sum_{j=1}^{n-k}\left[\binom{j+r-1}{j}B(n-j, r, k)/(n-j-1)!\right]. \tag{42}$$

Then

$$E(N_i) = n/(k+1) \tag{43}$$

and

$$\text{Var}(N_i) = [n^2k/(k+1)^2 - (n!/(k+1)!)A(n, r, k+1)]. \tag{44}$$

Further,

$$\text{Cov}(N_i, N_j) = -(1/k)\text{Var}(N_i) \qquad (i \neq j) \tag{45}$$

and

$$\text{Corr}(N_i, N_j) = -(1/k). \tag{46}$$

The marginal distribution of N_1 is:

$$Pr\left\{N_1 = n_1 \,\middle|\, \sum_{i=1}^{k+1} N_i = n, k, r\right\}$$

$$= (n)_{n_1}\binom{n_1 + r - 1}{n_1}B(n - n_1, r, k)/[(k+1)B(n, r, k+1)], \tag{47}$$

$$n_1 = 1, \ldots, n-k.$$

The joint distribution of the subset $(N_1, \ldots, N_m)$ of the N_i's is:

$$Pr\left\{N_1 = n_1, \ldots, N_m = n_m \,\middle|\, \sum_{i=1}^{k+1} N_i = n, k, r\right\}$$

$$= (n)_{n_0}\prod_{i=1}^{m}\binom{n_i + r - 1}{n_i}B(n - n_0, r, k - m + 1)/[(k+1)_m B(n, r, k+1)], \tag{48}$$

where $n_0 = n_1 + \cdots + n_m$, each n_i being a positive integer.

The conditional distribution of N_j, where $N_1, \ldots, N_{j-1}$ are fixed, is:

$$Pr\left\{N_j = n_j \middle| N_1 = n_1, \ldots, N_{j-1} = n_{j-1}, \sum_{i=1}^{k+1} N_i = n,\ k,\ r \text{ being positive integers}\right\}$$

$$= (n - n_0 + n_j)!\binom{n_j + r - 1}{n_j} B(n - n_0, r, k - j + 1)/\{(n - n_0)!(k - j + 2)$$

$$\times B(n - n_0 + n_j, r, k - j + 2)\}, \tag{49}$$

where $n_0 = n_1 + \cdots + n_j$ and $n_j = 1, \ldots, n - n_0 + n_j - k + j - 1$.

It is interesting to note that the distribution of the vector $\overline{N}$ in (41) is the same as that of the joint distribution of the independent random variables $N_1, \ldots, N_{k+1}$, each following a zero truncated negative binomial distribution with arbitrary parameters θ $(0 < \theta < 1)$ and r (a positive integer), subject to the condition $N_1 + \cdots + N_{k+1} = n$.

9. AN APPLICATION OF $L_{c,r}(n, k)$

Let $n > ck$ indistinguishable balls be distributed in rk cells constituting k groups of r cells each. Then the probability that j groups of cells are occupied with each group containing at least $c + 1$ balls is given by

$$P_{c,r}(j \mid n) = (k)_j L_{c,r}(n, j)/(rk)^{[n]}, \tag{50}$$

where $(k)_j = k(k - 1) \ldots (k - j + 1)$ and

$$(rk)^{[n]} = (rk)(rk + 1) \ldots (rk + n - 1).$$

Proof: The probability that j groups $g_1, \ldots, g_j$ contain $x_1, \ldots, x_j$ balls, respectively, with $x_1 + \cdots + x_j = n$ is given by

$$\binom{x_1 + r - 1}{x_1} \ldots \binom{x_j + r - 1}{x_j} \Big/ \binom{n + rk - 1}{n}. \tag{51}$$

Therefore, the probability that the groups $g_1, \ldots, g_j$ are occupied each containing at least $c + 1$ balls is given by

$$\sum \binom{x_1 + r - 1}{x_1} \ldots \binom{x_j + r - 1}{x_j} \Big/ \binom{n + rk - 1}{n} \tag{52}$$

where the summation is extended over all ordered j-tuples $(x_1, \ldots, x_j)$ of integers $x_i > c$, $i = 1, \ldots, j$ with $x_1 + \cdots + x_j = n$.

Now, from (10), (52), and noting that j groups out of k can be selected in $\binom{k}{j}$ ways, we obtain (50).

10. A CONDITIONAL MULTIVARIATE DISTRIBUTION INVOLVING GENERALIZED LAH NUMBERS

From (10), the joint distribution of $\overline{N} = (N_1, \ldots, N_k)$ (given $N_1 + \cdots + N_k + N_{k+1} = n$) is

$$Pr\left\{\overline{N} = \overline{n} \middle| \text{each } n_i > c,\ i = 1, \ldots, k,\ n > \sum_{i=1}^{k} n_i,\ k,\ c, \text{ and } r \text{ are positive integers}\right\}$$

$$= (n!/(k + 1)!)\prod_{i=1}^{k+1}\binom{n_i + r - 1}{n_i} \Big/ L_{c,r}(n, k), \tag{53}$$

where the mass points of $\overline{n}$ are given by the set

$$\left\{\overline{n} \,\middle|\, \text{each } n_i > c,\ n > \sum_{i=1}^{k} n_i,\ k,\ c, \text{ and } r \text{ are fixed positive integers}\right\}.$$

We note that (53) represents the joint distribution of $k + 1$ independent random variables $N_1, \ldots, N_{k+1}$ each following a c-truncated negative binomial distribution with arbitrary parameters θ $(0 < \theta < 1)$, r and c subject to the condition $N_1 + \cdots + N_{k+1} = n$.

Distribution (53) has properties analogous to those of distribution (41).

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