LETTER TO THE EDITOR

NOTE ON "REPRESENTING $\binom{2n}{n}$ AS A SUM OF SQUARES"

[Neville Robbins, *The Fibonacci Quarterly* 25, no. 1 (1987):29]

In addition to the theorems Dr. Robbins presented, it is the case that

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2. \tag{1}$$

Proof: In general, the coefficients of terms in a polynomial that is the product of two other polynomials is the convolution of the terms of the two-factor polynomials. In particular, the coefficients of the terms in the binomial expansion can be expressed by such a convolution:

$$\binom{p}{q} = \sum_{i=0}^{n} \binom{p-r}{i}\binom{r}{q-i}. \tag{2}$$

If we chose $r = q = n$, then $p = 2n$, and we get

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}, \tag{3}$$

which is obviously equivalent to (1).

Equation (2) is a rendering of the first form of the Vandermonde convolution (see [1]), with the term $\binom{n}{p}$ replaced by 1. Equation (3) is a particular case of that, with the substitutions noted.

Corollary: $n!$ can be written recursively not only as $n(n-1)!$, but also (for even n) as

$$n! = (n/2)!^2 \sum_{i=0}^{n/2} \binom{n/2}{i}^2. \tag{4}$$

Proof: This is made clear by rewriting the summation according to (1) above:

$$n! = (n/2)!^2 \binom{n}{n/2}. \tag{5}$$

We then expand the combination $\binom{n}{n/2}$ to give,

$$n! = (n/2)!^2 \frac{n!}{(n/2)!(n-n/2)!}, \tag{6}$$

which is fairly obviously an identity.

Reference

1. John Riordan. *Combinatorial Identities*. New York: Wiley & Sons, 1968, p. 15, Eq. (9), form 1.

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