TRANSPOSABLE INTEGERS IN ARBITRARY BASES*

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1. INTRODUCTION

Let k be a positive integer. The n-digit number $x = a_{n-1}a_{n-2} \ldots a_1a_0$ is called k-transposable if and only if

$$kx = a_{n-2}a_{n-3} \ldots a_0a_{n-1}. \tag{1}$$

Clearly x is 1-transposable if and only if all of its digits are equal. Thus, we assume $k > 1$.

Kahan has studied decadic k-transposable integers (see [1]); that is, numbers expressed in base 10. The numbers $x_1 = 142857$ and $x_2 = 285714$ are both 3-transposable:

$$3(142857) = 428571$$
$$3(285714) = 857142$$

Kahan has shown that decadic k-transposable numbers exist only when $k = 3$. Further, all 3-transposable integers are obtained by concatenating x_1 or x_2 m times, $m \geqslant 1$ [1]. In this paper we will study k-transposable integers for an arbitrary base g.

2. TRANSPOSABLE INTEGERS IN BASE g

Let x be an n-digit number expressed in base g; that is,

$$x = \sum_{i=0}^{n-1} a_i g^i$$

with $0 \leqslant a_i < g$ and $a_{n-1} \neq 0$. Then x will be k-transposable if and only if

$$kx = \sum_{i=0}^{n-2} a_i g^{i+1} + a_{n-1}. \tag{2}$$

Again we assume $k > 1$; further, we can assume that $k < g$, since $k \geqslant g$ would imply that kx has more digits than x. By rewriting (2), we see that the digits of x must satisfy the following equation:

$$(kg^{n-1} - 1)a_{n-1} = (g - k)\sum_{i=0}^{n-2} a_i g^i. \tag{3}$$

Let d be the greatest common divisor of $g - k$ and $kg^{n-1} - 1$, written

$$d = (g - k,\ kg^{n-1} - 1).$$

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Then the following lemma gives information about d.

Lemma: Let x be an n-digit k-transposable g-adic integer and let

$$d = (g - k,\ kg^{n-1} - 1).$$

Then d must satisfy the following:

(i) $(d, k) = 1$
(ii) $k < d$
(iii) $k^n \equiv 1 \pmod{d}$

Proof: Properties (i) and (iii) follow immediately from the definition of d.

To show (ii), suppose $d \leqslant k - 1$. Then, in (3), $(g - k)$ divides the left-hand side (LHS) as follows:

$$d \text{ divides } kg^{n-1} - 1 \quad \text{and} \quad \frac{g - k}{d} \text{ divides } a_{n-1}.$$

Thus,

$$\frac{kg^{n-1} - 1}{d} > \frac{(k - 1)g^{n-1}}{d} \geqslant g^{n-1} \text{ by the assumption.}$$

But, then, the LHS divided by $g - k$ has a g^{n-1} term, while the right-hand side (RHS) does not. Since $(d, k) = 1$, $k < d$.

We are now able to determine those g-adic numbers which are k-transposable for some k.

Theorem 1: There exists an n-digit g-adic k-transposable integer if and only if there exists an integer d which satisfies the following properties:

(i) $(d, k) = 1$
(ii) $k < d$
(iii) $d \mid g - k$
(iv) $k^n \equiv 1 \pmod{d}$

Proof: If x is k-transposable then, by the lemma, $d = (g - k, kg^{n-1} - 1)$ satisfies (i)-(iv).

To show the converse, we first observe that d divides $kg^{n-1} - 1$:

$$kg^{n-1} - 1 \equiv kk^{n-1} - 1 \equiv k^n - 1 \equiv 0 \pmod{d}.$$

We now define $x = \sum_{i=0}^{n-1} a_i g^i$ which satisfies (3). Let

$$a_{n-1} = \frac{g - k}{d}. \tag{4}$$

Since $k < d$, $(kg^{n-1} - 1)/d$ has no g^{n-1} term. Thus, $a_{n-2}, \ldots, a_0$ are well defined by the following equation:

$$\sum_{i=0}^{n-2} a_i g^i = \frac{kg^{n-1} - 1}{d}. \tag{5}$$

Note that (5) is obtained by dividing (3) by $g - k = d((g - k)/d)$.

For d satisfying (i)-(iv), we can actually find $[d/k]$ k-transposable integers. We will define

$$x_t = \sum_{i=0}^{n-1} b_{t,i} g^i, \text{ where } t = 1, \ldots, \left[\frac{d}{k}\right].$$

Let $b_{t,i}$ be given by

$$b_{t,n-1} = \left(\frac{g-k}{d}\right)t \tag{6}$$

and

$$\sum_{i=0}^{n-2} b_{t,i}g^i = \left(\frac{kg^{n-1}-1}{d}\right)t. \tag{7}$$

Note that in (7) the RHS has no g^{n-1} term since $kt \leqslant d$; thus, the $b_{t,i}$ are well defined.

We will shortly give an example to show how Theorem 1 is used to determine all k-transposable integers for a given g. We note here that the proof of Theorem 2 is a constructive one. The digits of k-transposable numbers are found using (6) and (7). We now show that almost all g have k-transposable integers.

Theorem 2: If $g = 5$ or $g \geqslant 7$, then there exists a k-transposable integer for some k. No k-transposable numbers exist for $g = 2, 3, 4, 6$.

Proof: Recall that $k > 1$. For the first part we must find k with the following properties:

$$2 \leqslant k < \frac{g}{2}$$

$$(k, g) = 1$$

If g is odd, let $k = 2$. Otherwise, if $g = 2h$, $h \geqslant 4$, choose

$$k = \begin{cases} h - 1 & \text{if } h \text{ is even,} \\ h - 2 & \text{if } h \text{ is odd.} \end{cases}$$

Now let $d = g - k$. Then, clearly, d satisfies (i)-(iii) of Theorem 1. Since $(d, k) = 1$ and $k < d$, there exists n with $k^n \equiv 1 \pmod{d}$. Hence, by Theorem 1, there is an n-digit g-adic k-transposable integer.

It is a straightforward matter to check that there are no k-transposable integers when $g = 2, 3, 4, 6$.

We now show that up to concatenation there are only a finite number of k-transposable integers for a given k, and hence a finite number for a given g.

Theorem 3: Suppose $x = \sum_{i=0}^{n-1} a_i g^i$ is a k-transposable integer. Let

$$d = (g - k, kg^{n-1} - 1)$$

and let N be the order of k in U_d, the group of units of Z_d. Then x equals some N-digit k-transposable integer concatenated n/N times.

Proof: Since $k^n \equiv 1 \pmod{d}$, n is a multiple of N. Let

$$x_t = \sum_{i=0}^{N-1} b_{t,i}g^i, \quad t = 1, \ldots, \left[\frac{d}{k}\right],$$

be the N-digit integers given by equations (6) and (7).

As shown in the proof of Theorem 1, $(g - k)/d$ divides a_{n-1} while d divides $kg^{n-1} - 1$. Thus,

$$a_{n-1} = \frac{g-k}{d} \cdot t = b_{t,N-i} \text{ for some } t \leqslant \left[\frac{d}{k}\right].$$

Now,

$$\sum_{i=0}^{n-2} a_i g^i = \left(\frac{kg^{n-1} - 1}{d}\right)t = g^{n-N}\left(\frac{kg^{N-1} - 1}{d}\right)t + \left(\frac{g^{n-N} - 1}{d}\right)t.$$

Hence,

$$a_{n-i} = b_{t,N-i},\ i = 2, \ldots, N,$$

since

$$\sum_{i=0}^{N-2} b_{t,i}\, g^i = \left(\frac{kg^{N-1} - 1}{d}\right)t.$$

But now we have

$$\left(\frac{g^{n-N} - 1}{d}\right)t = \left(\frac{g - k}{d}\right)tg^{n-N-1} + \left(\frac{kg^{n-N-1} - 1}{d}\right)t.$$

Thus,

$$a_{n-N-1} = \left(\frac{g - k}{d}\right)t = b_{t,N-1}$$

and

$$a_{n-N-i} = b_{t,N-i},\ i = 2, \ldots, N.$$

Continuing, we see that x equals x_t concatenated n/N times.

The N-digit numbers x_t are called basic k-transposable integers, since all others are obtained by concatenating these.

3. SOME EXAMPLES

We show how to determine all k-transposable integers for a given g by considering an example. By Theorem 3, we need only determine the basic k-transposable numbers.

Before beginning the example, we note that we need only consider $k < g/2$. By Theorem 1, $k < d$ and $d \mid g - k$; thus, $k \leqslant g/2$. Since $(d, k) = 1$, $k \neq g/2$.

Let $g = 9$: the possibilities for k, d, and N are given in the table.

k	$g - k$	d	N
2	7	7	3
3	6	–	–
4	5	5	2

When $k = 2$, there are $\left[\frac{d}{k}\right] = 3$, 2-transposable integers. These are found using (6) and (7):

$$b_{t,2} = t;$$

$$b_{t,1} \cdot 9 + b_{t,0} = \left(\frac{2 \cdot 9^2 - 1}{7}\right)t = 23t,\ t = 1, 2, 3.$$

Thus, the basic 2-transposable integers are 125, 251, 376. (Note that these numbers are expressed in base 9.) When $k = 4$, there is one 4-transposable integer, namely, 17.

It is possible that, for a given g and k, there will be more than one d which satisfies (i)-(iii) of Theorem 1. We illustrate this with an example. Suppose $g = 17$ and $k = 2$. Since $g - k = 15$, d can equal 3, 5, or 15. The 2-transposable integers for each case are given in the following table.

d	N	$\left[\frac{d}{k}\right]$	x		
3	2	1	$5\ \overline{11}$		
5	4	2	$3\ 6\ \overline{13}\ \overline{10}$	$6\ \overline{13}\ \overline{10}\ 3$	
			$1\ 2\ 4\ 9$	$4\ 9\ 1\ 2$	
15	4	7	$2\ 4\ 9\ 1$	$5\ \overline{11}\ 5\ \overline{11}$	$7\ \overline{15}\ \overline{14}\ \overline{12}$
			$3\ 6\ \overline{13}\ \overline{10}$	$6\ \overline{13}\ \overline{10}\ 3$	

Note that the 2-transposable integers corresponding to $d = 3, 5$ are included among those for $d = 15$, except that $5\ \overline{11}\ 5\ \overline{11}$ is not basic.

REFERENCE

1. Steven Kahan. "k-Transposable Integers." *Math. Magazine* 49, no. 1 (1976): 27-28.

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