MIXED PELL POLYNOMIALS

A. F. HORADAM
University of New England, Armidale, Australia

Bro. J. M. MAHON
Catholic College of Education, Sydney, Australia

(Submitted April 1985)

1. INTRODUCTION

Pell polynomials $P_n(x)$ are defined ([8], [13]) by

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1. \tag{1.1}$$

Pell-Lucas polynomials $Q_n(x)$ are likewise defined ([8], [13]) by

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Properties of $P_n(x)$ and $Q_n(x)$ can be found in [8] and [13], while convolution polynomials for $P_n(x)$ and $Q_n(x)$ are investigated in detail in [9].

The k^{th} *convolution sequence for Pell polynomials* $\{P_n^{(k)}(x)\}$, $n = 1,2,3,\ldots$, is defined in [9] by the equivalent expressions

$$P_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} P_i(x)P_{n+1-i}^{(k-1)}(x) & k \geqslant 1 \\ \sum_{i=1}^{n} P_i^{(1)}(x)P_{n+1-i}^{(k-2)}(x) & \begin{cases} P_n^{(0)}(x) = P_n(x) \\ P_0^{(k)}(x) = 0 \end{cases} \\ \ldots & \\ \sum_{i=1}^{n} P_i^{(m)}(x)P_{n+1-i}^{(k-1-m)}(x) & 0 \leqslant m \leqslant k-1 \end{cases} \tag{1.3}$$

for which the generating function is

$$(1 - 2xy - y^2)^{-(k+1)} = \sum_{n=0}^{\infty} P_{n+1}^{(k)}(x)y^n. \tag{1.4}$$

The k^{th} *convolution sequence for Pell-Lucas polynomials* $\{Q_n^{(k)}(x)\}$, $n = 1,2,3,\ldots$, is defined in [9] by

$$Q_n^{(k)}(x) = \sum_{i=1}^{n} Q_i(x)Q_{n+1-i}^{(k-1)}(x),\ k \geqslant 1,\ Q_n^{(0)}(x) = Q_n(x) \tag{1.5}$$

with similar equivalent expressions in (1.5) for $Q_n^{(k)}(x)$ to those in (1.3) for $P_n^{(k)}(x)$. [$Q_0^{(k)}(x) = 0$ if $k \geqslant 1$; $Q_0^{(0)}(x) = 2$.]

The generating function for Pell-Lucas convolution polynomials is

$$\left\{\frac{2x + 2y}{1 - 2xy - y^2}\right\}^{k+1} = \sum_{n=0}^{\infty} Q_{n+1}^{(k)}(x)y^n. \tag{1.6}$$

Explicit summation formulas for the k^{th} convolutions are

$$P_n^{(k)}(x) = \sum_{r=0}^{[(n-1)/2]} \binom{k+n-1-r}{k}\binom{n-1-r}{r}(2x)^{n-2r-1} \tag{1.7}$$

and

$$Q_n^{(k)}(x) = 2^{k+1}\sum_{r=0}^{n-1}\binom{k+1}{r}x^{k+1-r}P_{n-r}^{(k)}(x) \tag{1.8}$$

where, in the latter case, the Pell-Lucas convolutions are expressed in terms of Pell convolutions.

A result needed subsequently is:

$$nP_{n+1}^{(k)}(x) = 2(k+1)\{xP_n^{(k+1)}(x) + P_{n-1}^{(k+1)}(x)\}. \tag{1.9}$$

Some of the simplest convolution polynomials are set out in Table 1.

Table 1. Convolutions for $P_n^{(k)}(x)$, $Q_n^{(k)}(x)$, $k = 1, 2;\ n = 1, 2, 3, 4, 5$

	$n = 1$	2	3	4	5
$P_n^{(1)}(x)$	1	$4x$	$12x^2 + 2$	$32x^3 + 12x$	$80x^4 + 48x^2 + 3$
$Q_n^{(1)}(x)$	$4x^2$	$16x^3 + 8x$	$48x^4 + 40x^2 + 4$	$128x^5 + 144x^3 + 32x$	$320x^6 + 448x^4 + 156x^2 + 8$
$P_n^{(2)}(x)$	1	$6x$	$24x^2 + 3$	$80x^3 + 24x$	$240x^4 + 120x^2 + 6$
$Q_n^{(2)}(x)$	$8x^3$	$48x^4 + 24x^2$	$192x^5 + 168x^3 + 24x$	$640x^6 + 768x^4 + 216x^2 + 8$	$1920x^7 + 2880x^5 + 1220x^3 + 120x$

Worth noting are the facts that

$$C_n^k(ix) = i^nP_{n+1}^{(k-1)}(x) \qquad (i = \sqrt{-1}), \tag{1.10}$$

where $C_n^k(x)$ is the *Gegenbauer polynomial* of degree n and order k [12], and

$$P_{n+1}^{(k)}(x) = P_n(2, x, -1, -(k+1), 1), \tag{1.11}$$

in which the right-hand side is a special case of the *generalized Humbert polynomial* $P_n(m, x, y, p, C)$ defined [3] by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n \qquad (m \geqslant 1). \tag{1.12}$$

Pell-Lucas convolution polynomials $Q_n^{(k)}(x)$ can be expressed in terms of the complex Gegenbauer polynomials by a complicated formula, but they are not expressible as specializations of generalized Humbert polynomials [cf. (1.6) and (1.12)].

Specializations of $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ of interest to us occur when $x = 1$, giving the convolution sequences for *Pell numbers* and *Pell-Lucas numbers*. If x is replaced by $\frac{1}{2}x$, the sequence of *Fibonacci polynomial convolutions* and the sequence of *Lucas convolution polynomials* arise; in this case, putting $x = 1$ gives convolution sequences for *Fibonacci numbers* and for *Lucas numbers*.

The chief object of this paper is not to concentrate on $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$, but to examine convolution polynomials when $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ are combined together. This will lead to the concept of "mixed Pell convolutions" and of a convolution of convolutions.

2. MIXED PELL CONVOLUTIONS

Let us introduce the *mixed Pell convolution* $\pi_n^{(a,b)}(x)$ in which

(i) $a + b \geqslant 1$

(ii) $\pi_n^{(0,0)}(x)$ is not defined.

Let

$$\sum_{n=0}^{\infty} \pi_{n+1}^{(a,b)}(x) y^n = \frac{(2x+2y)^b}{(1-2xy-y^2)^{a+b}} \tag{2.1}$$

$$= (2x+2y)^{b-j} \frac{1}{(1-2xy-y^2)^{a+b-j}} \left(\frac{2x+2y}{1-2xy-y^2}\right)^j$$

$$= (2x+2y)^{b-j}\left(\sum_{n=0}^{\infty} \pi_{n+1}^{(a+b-j,j)}(x) y^n\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{b-j} \binom{b-j}{i} (2x)^{b-j-i} 2^i \pi_{n+1-i}^{(a+b-j,j)}(x)\right) y^n$$

whence

$$\pi_{n+1}^{(a,b)}(x) = 2^{b-j} \sum_{i=0}^{b-j} \binom{b-j}{i} x^{b-j-i} \pi_{n+1-i}^{(a+b-j,j)}(x). \tag{2.2}$$

Put $j = 1$ in (2.2). Then

$$\pi_{n+1}^{(a,b)}(x) = 2^{b-1} \sum_{i=0}^{b-1} \binom{b-1}{i} x^{b-1-i} \pi_{n+1-i}^{(a+b-1,1)}(x) \tag{2.3}$$

Special cases of (2.1) occur when $a = 0$, and when $b = 0$.

Thus, for $b = 0$, and $a = k$, (1.4) and (2.1) show that, with $n + 1$ replaced by n,

$$\pi_n^{(k,0)}(x) = P_n^{(k-1)}(x), \tag{2.4}$$

i.e.,

$$\pi_n^{(1,0)}(x) = P_n(x) \text{ by (1.3)}, \quad \pi_n^{(2,0)}(x) = P_n^{(1)}(x).$$

On the other hand, when $a = 0$ and $b = k$, (1.6) and (2.1) yield

$$\pi_n^{(0,k)}(x) = Q_n^{(k-1)}(x), \tag{2.5}$$

i.e.,

$$\pi_n^{(0,1)}(x) = Q_n(x) \text{ by (1.5)}, \quad \pi_n^{(0,2)}(x) = Q_n^{(1)}(x).$$

Now let $j = 0$ in (2.2). Hence, by (2.4), with $n + 1$ replaced by n,

$$\pi_n^{(a,b)}(x) = 2^b \sum_{i=0}^{b} \binom{b}{i} x^{b-i} P_{n-i}^{(a+b-1)}(x). \tag{2.6}$$

An explicit formulation for $\pi_n^{(a,b)}(x)$ could then be given by substituting for $P_{n-i}^{(a+b-1)}(x)$ from (1.7).

From (2.1), with (1.4) and (1.6), it is seen that

$$\pi_n^{(a,b)}(x) = \sum_{i=1}^{n} P_i^{(a-1)}(x) Q_{n+1-i}^{(b-1)}(x) \qquad (a \geqslant 1,\ b \geqslant 1). \tag{2.7}$$

Let us differentiate both sides of (2.1) w.r.t. y. Then

$$\sum_{n=0}^{\infty} n\pi_{n+1}^{(a,b)}(x) y^{n-1} = 2b \sum_{n=0}^{\infty} \pi_{n+1}^{(a+1,b-1)}(x) y^n + (a+b) \sum_{n=0}^{\infty} \pi_{n+1}^{(a,b+1)}(x) y^n,$$

whence

$$n\pi_{n+1}^{(a,b)}(x) = 2b\pi_n^{(a+1,b-1)}(x) + (a+b)\pi_n^{(a,b+1)}(x). \tag{2.8}$$

From the identity

$$\frac{(2x+2y)^b}{(1-2xy-y^2)^{a+b}} \cdot \frac{(2x+2y)^a}{(1-2xy-y^2)^{b+a}} = \frac{(2x+2y)^{a+b}}{(1-2xy-y^2)^{2a+2b}}$$

we derive a *convolution of convolutions*

$$\pi_n^{(a+b,\,a+b)}(x) = \sum_{i=1}^{n} \pi_i^{(a,\,b)}(x)\pi_{n+1-i}^{(b,\,a)}(x). \tag{2.9}$$

So, when $b = a$,

$$\pi_n^{(2a,\,2a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,\,a)}(x)\pi_{n+1-i}^{(a,\,a)}(x). \tag{2.10}$$

From (2.9), when $b = 0$,

$$\pi_n^{(a,\,a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,\,0)}(x)\pi_{n+1-i}^{(0,\,a)}(x) = \sum_{i=1}^{n} P_i^{(a-1)}(x)Q_{n+1-i}^{(a-1)}(x) \tag{2.11}$$

on using (2.4) and (2.5). [Cf. (2.7) also for $b = a$.]

Putting $b = a$ in (2.8) leads to

$$n\pi_{n+1}^{(a,\,a)}(x) = 2a\pi_n^{(a+1,\,a-1)}(x) + 2a\pi_n^{(a,\,a+1)}(x). \tag{2.12}$$

Combining (2.9) and (2.12), we have

$$2a\{\pi_n^{(a+1,\,a-1)}(x) + \pi_n^{(a,\,a+1)}(x)\} = n\sum_{i=1}^{n} \pi_i^{(a,\,0)}(x)\pi_{n+1-i}^{(0,\,a)}(x). \tag{2.13}$$

Equations (2.5) and (2.6), in which $a = 0$ and $b = k + 1$, give

$$\pi_n^{(0,\,k+1)}(x) = Q_n^{(k)}(x) = 2^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}P_{n-i}^{(k)}(x)x^{k+1-i} \tag{2.14}$$

as in (1.8).

Next, put $b = 0$, $a = k$ in (2.8) to get

$$\begin{aligned}
\pi_n^{(k,\,1)}(x) &= \frac{n}{k}\,\pi_{n+1}^{(k,\,0)}(x) = \sum_{i=1}^{n} P_i^{(k-1)}(x)Q_{n+1-i}(x) && \text{by (2.7)} \qquad (2.15)\\
&= \frac{n}{k}\,P_{n+1}^{(k-1)}(x) && \text{by (2.4)}\\
&= 2xP_n^{(k)}(x) + 2P_{n-1}^{(k)}(x) && \text{by (1.9).}
\end{aligned}$$

To exemplify some of the above results, we write down alternative expressions for $\pi_3^{(2,\,2)}(x)$.

We have

$$\begin{aligned}
\pi_3^{(2,\,2)}(x) &= 4\{x^2P_3^{(3)}(x) + 2xP_2^{(3)}(x) + P_1^{(3)}(x)\} && \text{by (2.6)}\\
&= P_1^{(1)}(x)Q_3^{(1)}(x) + P_2^{(1)}(x)Q_2^{(1)}(x) + P_3^{(1)}(x)Q_1^{(1)}(x) && \text{by (2.7)}\\
&\left\{\begin{aligned} &= 2\{x\pi_3^{(3,\,1)}(x) + \pi_2^{(3,\,1)}(x)\} \\ &= 2\{x(3/3)P_4^{(2)}(x) + (2/3)P_3^{(2)}(x)\}\end{aligned}\right. && \begin{aligned}&\text{by (2.3)}\\ &\text{by (2.15)}\end{aligned}\\
&= \pi_1^{(2,\,0)}(x)\pi_3^{(0,\,2)}(x) + \pi_2^{(2,\,0)}(x)\pi_2^{(0,\,2)}(x) + \pi_3^{(2,\,0)}(x)\pi_1^{(0,\,2)}(x) && \ldots\ \text{by (2.11)}\\
&= 160x^4 + 80x^2 + 4
\end{aligned}$$

on using Table 1 and $P_1^{(3)}(x) = 1$, $P_2^{(3)}(x) = 8x$, and $P_3^{(3)}(x) = 40x^2 + 4$. Observe that the second and fifth lines of the chain of equalities above are the same, by virtue of (2.4) and (2.5).

Some interesting results for particular values of a and b may be found. For example, with $a = 0$, $b = 2$, we have, by (2.5) and (2.8),

$$nQ_{n+1}^{(1)}(x) = 4\pi_n^{(1,1)}(x) + 2Q_n^{(2)} = 4(1 + x^2)\pi_n^{(2,1)} + Q_n^{(2)}$$

on rearranging in another way the terms in the differentiation of (2.1). [For instance, when $n = 2$, the common value is $90x^4 + 80x^2 + 8$ on using

$$P_3^{(1)}(x) = \pi_2^{(2,1)}(x) \qquad \text{by (2.15)},$$

and Table 1.]

Thus,

$$Q_n^{(2)}(x) = 4(1 + x^2)\pi_n^{(2,1)}(x) - \pi_n^{(1,1)}(x).$$

Using

$$\pi_n^{(1,1)}(x) = nP_{n+1}(x) = \sum_{i=1}^{n} P_i(x)Q_{n+1-i}(x), \tag{2.16}$$

from (2.15) and (1.3), we find that the simplest values of $\pi_n^{(1,1)}(x)$ are:

$$\begin{cases} \pi_1^{(1,1)}(x) = 2x, \quad \pi_2^{(1,1)}(x) = 8x^2 + 2, \quad \pi_3^{(1,1)}(x) = 24x^3 + 12x \\ \pi_4^{(1,1)}(x) = 64x^4 + 48x^2 + 4, \quad \pi_5^{(1,1)}(x) = 160x^5 + 160x^3 + 30x \ldots \end{cases}$$

Theoretically, one may obtain a Simson-type analogue for the mixed convolution function $\pi_n^{(a,b)}(x)$. However, the task is rather daunting, so we content ourselves with the Simson formula in the simple instance when $a = b = 1$.

Computation, with the aid of (2.16) produces

$$\pi_{n+1}^{(1,1)}(x)\pi_{n-1}^{(1,1)}(x) - (\pi_n^{(1,1)}(x))^2 = (-1)^{n+1}(n^2 - 1) - P_{n+1}^2(x) \tag{2.17}$$

(both sides being equal to $-16x^4 - 8x^2 - 4$ when, say, $n = 2$).

3. MISCELLANEOUS RESULTS

A. Pell Convolutions

Two results given in [3] are worth relating to convolution polynomials.

First, apply (1.11) to [3, (3.10)]. Then

$$P_{n+1}^{(k)}(x) = \sum_{i_1 + i_2 + \cdots + i_j = n} P_{i_1+1}(x)P_{i_2+1}(x) \ldots P_{i_j+1}(x) \tag{3.1}$$

in our system of polynomials. Observe the restriction on the summation. Putting $k = 2$ and $n = 2$, say, gives, on applying (1.3) the appropriate number of times,

$$\begin{aligned} P_3^{(2)}(x) &= P_1(x)P_2(x)P_2(x) + P_2(x)P_1(x)P_2(x) + P_2(x)P_2(x)P_1(x) \\ &\quad + P_1(x)P_1(x)P_3(x) + P_1(x)P_3(x)P_1(x) + P_3(x)P_1(x)P_1(x) \\ &= 24x^2 + 3 \end{aligned}$$

which is precisely the summation expansion in (3.1). We may think of the ordered subscripts in each three-term product of the sum as a solution-set of $x + y + z = 5$ for nonnegative integers.

Second, suppose we wish to expand a given Fibonacci polynomial, say $F\ (x)$, in terms of Pell polynomials (an example of a well-known type of problem in classical analysis—see [2]).

Using notation in [3, (6.9), (6.10)], we have

$$F_5(x) = x^4 + 3x^2 + 1 = \sum_{n=0}^{4} A_n x^n \tag{3.2}$$

i.e.,

$$A_0 = 1,\ A_1 = 0,\ A_2 = 3,\ A_3 = 0,\ A_4 = 1, \tag{3.3}$$

whence

$$F_5(x) = \sum_{n=0}^{4} V_n P_{n+1}(x), \tag{3.4}$$

where

$$V_n = \sum_{j=0}^{[(4-n)/2]} (-1)^n \frac{\binom{-n-1-j}{j}}{\binom{-1}{n+2j}} \cdot \frac{n+1}{n+1+j} \cdot \frac{A_{n+2j}}{2^{n+2j}}. \tag{3.5}$$

Expanding (3.5) and using (3.3), we calculate

$$V_0 = A_0 - \frac{A_2}{4} + \frac{A_4}{8},\quad V_1 = -\left(\frac{A_1}{2} - \frac{A_3}{4}\right) = 0,\quad V_2 = \frac{A_2}{4} - \frac{3A_4}{16},$$

$$V_3 = -\frac{A_3}{8} = 0,\quad V_4 = \frac{A_4}{16}$$

whence the right-hand side of (3.4) simplifies to (3.2) on using (1.1) to obtain appropriate Pell polynomials. Thus,

$$F_5(x) = \frac{3}{8} P_1(x) + \frac{9}{16} P_3(x) + \frac{1}{16} P_5(x).$$

Again,

$$P_5^{(1)}(x) = P_1(x) - 3P_3(x) + 5P_5(x) \qquad (= 80x^4 + 48x^2 + 3)$$

on paralleling the calculations above.

Computations involving Pell convolution polynomials $P_n^{(k)}(x)$ for $k \geqslant 1$ could be effected in a similar manner.

B. Even and Odd Pell Convolutions

Let us now introduce $^*P_n^{(1)}(x)$, the *first convolution of even Pell polynomials*, i.e., of Pell polynomials with even subscripts.

Consider

$$\sum_{n=0}^{\infty} P_{2n+2}(x) y^n = \frac{2x}{1 - Q_2(x)y + y^2}, \tag{3.6}$$

where $Q_2(x) = 4x^2 + 2$ [by (1.2)] and the nature of the generating function is determined by the recurrence relation for the Pell polynomials with even subscripts, which is obtained by a repeated application of (1.1), namely

$$P_n(x) = (4x^2 + 2)P_{n-2}(x) - P_{n-4}(x). \tag{3.7}$$

Then

$$\left(\sum_{n=0}^{\infty} P_{2n+2}(x) y^n\right)^2 = \frac{4x^2}{(1 - Q_2(x)y + y^2)^2} \tag{3.8}$$

that is,

$$\sum_{n=0}^{\infty} {}^*P_{n+1}^{(1)}(x)y^n = \frac{4x^2}{(1 - Q_2(x)y + y^2)^2}, \tag{3.9}$$

where

$${}^*P_n^{(1)}(x) = \sum_{i=1}^{n} P_{2i}(x)P_{2n+2-2i}(x). \tag{3.10}$$

Some expressions for these convolutions are:

$$\begin{cases} {}^*P_1^{(1)}(x) = P_2(x)P_2(x) = 4x^2 \\ {}^*P_2^{(1)}(x) = P_2(x)P_4(x) + P_4(x)P_2(x) = 32x^4 + 16x^2 \\ {}^*P_3^{(1)}(x) = P_2(x)P_6(x) + P_4(x)P_4(x) + P_6(x)P_2(x) \\ \qquad\qquad = 192x^6 + 192x^4 + 40x^2 \end{cases}$$

Properties similar to those given in [9; (4.3), (4.4), (4.5), ...] may be obtained. Analogous to [9, (4.3)], for instance, we have the basic recursion-type relation

$${}^*P_n^{(1)}(x) - Q_2(x){}^*P_{n-1}^{(1)}(x) + {}^*P_{n-2}^{(1)}(x) = P_2(x)P_{2n}(x). \tag{3.11}$$

If we differentiate in (3.6) w.r.t. y and compare the result with (2.4), we deduce the analogue of [9, 4.4)]:

$$2nxP_{2n+2}(x) = Q_2(x){}^*P_n^{(1)}(x) - 2{}^*P_{n-1}^{(1)}(x). \tag{3.12}$$

Experimentation has also been effected with convolutions of *odd* Pell polynomials (i.e., Pell polynomials with odd subscripts), with convolutions for Pell polynomials having subscripts, say, of the form $3m$, $3m + 1$, $3m + 2$, and generally with convolutions for Pell polynomials having subscripts of the form $rm + k$.

For the odd-subscript Pell polynomials, the recurrence relation is of the same form as that in (3.7). Indeed, $x = 1$ gives the recurrence

$$P_n = 6P_{n-2} - P_{n-4},$$

which is valid for sequences of Pell numbers with even subscripts or odd subscripts. Compare the situation for sequences of Fibonacci numbers with even subscripts or odd subscripts for which the recurrence is

$$F_n = 3F_{n-2} - F_{n-4}.$$

Other possibilities include convolving even and odd Pell polynomials, and powers of Pell polynomials.

Generalizing the above work to results for n^{th} convolutions is a natural extension.

Of course, investigations involving Pell polynomials automatically include considerations of cognate work on Pell-Lucas polynomials, and of a study of mixed convolutions of artibrary order, as for $\pi_n^{(a,b)}(x)$.

C. Further Developments

Among other possible developments of our ideas, we mention the generation of $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ by rising diagonals of a Pascal-type array as was done in [8] for $P_n(x)$ and $Q_n(x)$. Work on this aspect is under way.

A variation of this approach is an examination of the polynomials produced by the rising (and descending) diagonals of arrays whose rows are the coefficients of powers of x in $P_n^{(k)}(x)$, where $n = 1, 2, 3, \ldots$, for a given k. Such a treatment as this has been done in [6], [7], and [10] for Chebyshev, Fermat, and Gegenbauer polynomials.

Another problem which presents itself is a discussion of the convolutions of Pell polynomials and *Pell-Jacobsthal polynomials* which might be defined by the recurrence relation

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x) \qquad J_0(x) = 0,\ J_1(x) = 1. \tag{3.13}$$

Evidently, one can proceed *ad infinitum, ad nauseam*! Convolution work on on Fibonacci polynomials and *Jacobsthal polynomials*, defined in [5] and [11], is summarized in [14]. The chapter on Convolutions in [14], a thesis dedicated to the mathematical research of the late Verner E. Hoggatt, Jr., contains much other information on convolution arrays for well-known sequences, such as the *Catalan sequence*, studied by Hoggatt and his associates.

D. Case $x = 1$

Following procedures established in [1] and [4] for Fibonacci number convolutions, we may demonstrate *inter alia* the results:

$$8P_n^{(1)} = (3n + 1)P_{n+1} - (n + 1)P_{n-1}; \tag{3.14}$$

$$8P_n^{(1)} = nQ_{n+1} + 2P_n; \tag{3.15}$$

$$P_{n+4}^{(1)} = 4P_{n+3}^{(1)} - 2P_{n+2}^{(1)} - 4P_{n+1}^{(1)} - P_n^{(1)}; \tag{3.16}$$

$$Q_{n-1}P_n^{(1)} - Q_{n+1}P_{n-2}^{(1)} = 2P_n^2; \tag{3.17}$$

$$\begin{vmatrix} P_{n+3}^{(1)} & P_{n+2}^{(1)} & P_{n+1}^{(1)} & P_n^{(1)} \\ P_{n+2}^{(1)} & P_{n+1}^{(1)} & P_n^{(1)} & P_{n-1}^{(1)} \\ P_{n+1}^{(1)} & P_n^{(1)} & P_{n-1}^{(1)} & P_{n-2}^{(1)} \\ P_n^{(1)} & P_{n-1}^{(1)} & P_{n-2}^{(1)} & P_{n-3}^{(1)} \end{vmatrix} = +1. \tag{3.18}$$

Clearly, all the work in this paper for k^{th} convolutions of the Pell and Pell-Lucas polynomials can be specialized for Pell and Pell-Lucas numbers.

REFERENCES

1. G. E. Bergum & V. E. Hoggatt, Jr. "Limits of Quotients for the Convolved Fibonacci Sequence and Related Sequences." *The Fibonacci Quarterly* **15**, no. 2 (1977):113-16.
2. P. F. Byrd. "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers." *The Fibonacci Quarterly* **1**, no. 1 (1963):16-29.
3. H. W. Gould. "Inverse Series Relations and Other Expansions Involving Humbert Polynomials." *Duke Math. J.* **32**, no. 4 (1965):697-712.
4. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Fibonacci Convolution Sequences." *The Fibonacci Quarterly* **15**, no. 2 (1977):117-22.
5. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Convolution Arrays for Jacobsthal and Fibonacci Polynomials." *The Fibonacci Quarterly* **16**, no. 5 (1978):385-402.
6. A. F. Horadam. "Polynomials Associated with Chebyshev Polynomials of the First Kind." *The Fibonacci Quarterly* **15**, no. 3 (1977):255-57.

7. A. F. Horadam. "Chebyshev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly* 17, no. 4 (1979):378-83.
8. A. F. Horadam & (Bro.) J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* 23, no. 1 (1985):7-20.
9. A. F. Horadam & (Bro.) J. M. Mahon. "Convolutions for Pell Polynomials." In *Fibonacci Numbers and Their Applications*. Dordrecht, The Netherlands: D. Reidel Publishing Company, 1986.
10. A. F. Horadam & S. Pethe. "Polynomials Associated with Gegenbauer Polynomials." *The Fibonacci Quarterly* 19, no. 5 (1981):393-98.
11. E. Jacobsthal. "Fibonaccische Polynome und Kreisteilungsgleichungen." *Berliner Mathematische Gesellschaft. Sitzungsberichte* 17 (1919-20):43-57.
12. W. Magnus, F. Oberhettinger, & R. P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Berlin: Springer-Verlag, 1966.
13. (Bro.) J. M. Mahon. "Pell Polynomials." M.A. (Hons.) Thesis, University of New England, Australia, 1984.
14. J. Spraggon. "Special Aspects of Combinatorial Number Theory." M.A. (Hons.) Thesis, University of New England, Australia, 1982.

◆◇◆◇◆