THE RECIPROCAL OF THE BESSEL FUNCTION $J_k(z)$

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1. INTRODUCTION

For $k = 0, 1, 2, \ldots,$ let $J_k(z)$ be the Bessel function of the first kind. Put

$$f_k(z) = J_k(2\sqrt{z})/z^{k/2} = \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!(m+k)!} \tag{1.1}$$

and define the polynomial $u_m(k;\, x)$ by means of

$$k!f_k(xz)/f_k(z) = \sum_{m=0}^{\infty} u_m(k;\, x)\frac{z^m}{m!(m+k)!}, \tag{1.2}$$

Certain congruences for $w_m(x) = u_m(0;\, x)$ and the integers $w_m = w_m(0)$ were derived by Carlitz [3] in 1955, and an interesting application was presented.

The purpose of the present paper is to extend Carlitz's results to the polynomials $u_m(k;\, x)$ and the rational numbers $u_m(k) = u_m(k;\, 0)$.

In particular, we show in §§3 and 4 that, if p is a prime number, $p > 2k$, and

$$m = c_0 + c_1 p + c_2 p^2 + \cdots \quad (0 \leqslant c_0 < p - 2k)$$
$$(0 \leqslant c_i < p \text{ for } i > 0), \tag{1.3}$$

then

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{c_1} w_{c_2} \;\ldots\; \pmod{p}, \tag{1.4}$$

$$u_m(k;\, x) \equiv u_{c_0}(k;\, x) \cdot w_{c_1}^{p}(x) \cdot w_{c_2}^{p^2}(x) \;\ldots\; \pmod{p}. \tag{1.5}$$

In §5, we prove more general congruences of this type. In §6, applications of these general results are given. Finally, in §7, we examine in more detail the positive integers $u_n(1)$.

2. PRELIMINARIES

Throughout the paper, we use the notation $w_m(x) = u_m(0;\, x)$ and $w_m = w_m(0)$.

In the proofs of Theorems 1-6, we use the divisibility properties of binomial coefficients given in the lemmas below. These lemmas follow from well-known theorems of Kummer [4] and Lucas [5].

Lemma 1: If p is a prime number, then

$$\binom{mp}{rp} \equiv \binom{m}{r} \pmod{p}.$$

Also, if $p - 2k > s \geqslant 0$, then, for $j = s + 1,\, s + 2,\, \ldots,\, p - 1$,

$$\binom{np + s + k}{rp + j + k}\binom{np + s + k}{rp + j} \equiv 0 \pmod{p}.$$

Lemma 2: Suppose p is a prime number and

$$n = n_0 + n_1 p + \cdots + n_j p^j \quad (0 \leqslant n_i < p),$$
$$r = r_0 + r_1 p + \cdots + r_j p^j \quad (0 \leqslant r_i < p),$$

If, for some fixed i, we have $r_i > n_i$ and $r_{i+v} \geqslant n_{i+v}$ for $v = 1, \ldots, t-1$, then

$$\binom{n}{r} \equiv 0 \pmod{p^t}.$$

Lemma 3: Let p be a prime number, $p > 2k$. Then

$$\binom{n+k}{r+k}\binom{n+k}{r}\Big/\binom{n+k}{k}$$

is integral (mod p) for $r = 0, 1, \ldots, n$. Also

$$\binom{mp}{rp+k}\Big/\binom{mp}{k} \equiv \binom{m-1}{r} \pmod{p},$$

$$\binom{mp}{rp-k}\Big/\binom{mp}{k} \equiv \binom{m-1}{r-1} \pmod{p}.$$

3. THE NUMBERS $u_m(k)$

We first note that the numbers $u_m(k)$ were introduced in [2], where Carlitz showed they cannot satisfy a certain type of recurrence formula.

It follows from (1.2) that

$$\{f_k(z)\}^{-1} = \sum_{m=0}^{\infty} u_m(k) \frac{z^m}{m!(m+k)!}. \tag{3.1}$$

Thus, we have

$$u_0(k) = u_1(k) = (k!)^2,$$
$$u_2(k) = (k!)^2(k+3)/(k+1),$$
$$u_3(k) = (k!)^2(k^2+8k+19)/(k+1)^2,$$

and

$$\sum_{r=0}^{m} (-1)^r \binom{m+k}{r+k}\binom{m+k}{r} u_r(k) = 0 \quad (m > 0). \tag{3.2}$$

It follows from (3.2) and Lemma 3 that if p is a prime number, $p \geqslant 2k$, then the numbers $u_m(k)$ are integral (mod p); in particular, $u_n(0)$ and $u_n(1)$ are positive integers for $n = 0, 1, 2, \ldots$.

Theorem 1: If p is a prime number and if $0 \leqslant s < p - 2k$, then

$$u_{np+s}(k) \equiv u_s(k) \cdot w_n \pmod{p}. \tag{3.3}$$

Proof: We use induction on the total index $np + s$. If $np + s = 0$, (3.3) holds since $w_0 = 1$. Assume (3.3) holds for all $rp + j < np + s$, with $j < p - 2k$. We then have, by (3.2),

$$(-1)^{n+s+1}\binom{s+k}{s} u_{np+s}(k) \equiv \sum_{r=0}^{n-1} \sum_{j=0}^{s} (-1)^{j+r} \binom{s+k}{j+k}\binom{s+k}{j}\binom{n}{r}^2 u_{rp+j}(k)$$
$$+ (-1)^n \sum_{j=0}^{s-1} (-1)^j \binom{s+k}{j+k}\binom{s+k}{j} u_{np+j}(k)$$

$$\equiv \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}^2 w_r \cdot \sum_{j=0}^{s}(-1)^j\binom{s+k}{j+k}\binom{s+k}{j}u_j(k)$$

$$+ (-1)^n w_n \sum_{j=0}^{s-1}(-1)^j\binom{s+k}{j+k}\binom{s+k}{j}u_j(k)$$

$$\equiv \begin{cases} 0 + (-1)^{n+s+1}\binom{s+k}{s} w_n u_s(k) \pmod{p} & \text{if } s > 0, \\ (-1)^{n+1} w_n u_0(k) \pmod{p} & \text{if } s = 0. \end{cases}$$

We see that (3.3) follows, and the proof is complete.

Corollary (Carlitz): With the hypotheses of Theorem 1 and with m defined by (1.3) with $k = 0$,

$$w_m \equiv w_{c_0} w_{c_1} w_{c_2} \ \ldots \pmod{p}.$$

Corollary: With the hypotheses of Theorem 1 and with m defined by (1.3),

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{c_1} w_{c_2} \ \ldots \pmod{p}.$$

Theorem 2: If p is a prime number, $p > 2k$, then

$$u_{np-k}(k) \equiv (-1)^k u_0(k) \cdot w_n \pmod{p}.$$

Proof: The proof is by induction on n. For $n = 1$ we have, by (3.1),

$$(-1)^k u_{p-k}(k) \equiv \sum_{r=0}^{p-k-1}(-1)^r\binom{p}{r+k}\binom{p}{r}u_r(k)\Big/\binom{p}{k}$$

$$\equiv u_0(k) \equiv u_0(k) \cdot w_1 \pmod{p}.$$

Theorem 2 is therefore true for $n = 1$; assume it is true for $n = 1, \ldots, s - 1$. Then

$$(-1)^{s+k+1}u_{sp-k}(k) \equiv \sum_{r=0}^{sp+k-1}(-1)^r\binom{sp}{r+k}\binom{sp}{r}u_r(k)\Big/\binom{sp}{k}$$

$$\equiv \sum_{r=0}^{s-1}(-1)^r\binom{sp}{rp+k}\binom{sp}{rp}u_{rp}(k)\Big/\binom{sp}{k}$$

$$+ \sum_{r=1}^{s-1}(-1)^{r-k}\binom{sp}{rp}\binom{sp}{rp-k}u_{rp-k}(k)\Big/\binom{sp}{k}$$

$$\equiv \sum_{r=0}^{s-1}(-1)^r\binom{s}{r}\binom{s-1}{r}u_0(k)w_r + \sum_{r=1}^{s-1}(-1)^r\binom{s}{r}\binom{s-1}{r-1}u_0(k)w_r$$

$$\equiv u_0(k)\sum_{r=0}^{s-1}(-1)^r\binom{s}{r}^2 w_r \equiv (-1)^{s-1}u_0(k)w_s \pmod{p}.$$

This completes the proof of Theorem 2.

If m is defined by (1.3) with $c_0 = p - k$, and if $c_i = p - 1$ for $1 \leqslant i \leqslant j - 1$ with $c_j < p - 1$, then Theorem 2 says

$$u_m(k) \equiv u_{c_0}(k) \cdot w_{1+c_j} w_{c_{j+1}} w_{c_{j+2}} \ \ldots \pmod{p}.$$

In particular, if $p > 2k$, and $n = p^t - k$,

$$u_n(k) \equiv u_{p-k}(k) \equiv (-1)^k u_0(k) \equiv (-1)^k (k!)^2 \pmod{p}.$$

4. THE POLYNOMIALS $u_m(k;\ x)$

We now consider the polynomials $u_m(k;\ x)$ defined by (1.2). It is clear that

$$u_m(k;\ 0) = u_m(k), \quad u_m(k,\ 1) = 0 \quad (m > 0).$$

Also, it follows from (1.1) and (1.2) that

$$\binom{m+k}{k} u_m(k;\ x) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m+k}{r+k} \binom{m+k}{r} u_r(k) x^{m-r}. \tag{4.1}$$

Theorem 3: If p is a prime number and if $0 \leqslant s < p - 2k$, then

$$u_{np+s}(k;\ x) \equiv u_s(k;\ x) \cdot w_{np}(x) \pmod{p}. \tag{4.2}$$

Proof: The proof is by induction on the total index $np + s$. We first note that

$$u_0(k;\ x) \equiv u_0(k;\ x) \cdot w_0(x) \pmod{p},$$

since $w_0(x) = 1$.

Assume (4.2) is true for all $rp + j < np + s$ with $0 \leqslant j < p - 2k$. Then, by (4.1) and (3.3),

$$\binom{s+k}{s} u_{np+s}(k;\ x) \equiv \sum_{r=0}^{np+s} (-1)^{n-s-r} \binom{np+s+k}{r} \binom{np+s+k}{r+k} u_r(k) x^{np+s-r}$$

$$\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} \binom{np+s+k}{rp+j} \binom{np+s+k}{rp+j+k} (-1)^{n+s+j+r} u_{rp+j}(k) x^{np-rp+s-j}$$

$$\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} \binom{n}{r}^2 \binom{s+k}{j} \binom{s+k}{j+k} (-1)^{n+s+j+r} w_r u_j(k) x^{np-rp+s-j}$$

$$\equiv \sum_{j=0}^{s} \binom{s+k}{j} \binom{s+k}{j+k} (-1)^{s+j} u_j(k) x^{s-j} \cdot \sum_{r=0}^{n} \binom{n}{r}^2 (-1)^{n-r} w_r x^{np-rp}$$

$$\equiv \binom{s+k}{k} u_s(k;\ x) \cdot w_n(x^p) \equiv \binom{s+k}{s} u_s(k;\ x) \cdot w_{np}(x) \pmod{p}.$$

This completes the proof of Theorem 3. We note that Theorem 1 was used in the proof.

Corollary (Carlitz): With the hypotheses of Theorem 3 and with m defined by (1.3) with $k = 0$,

$$w_m(x) \equiv w_{c_0}(x) \cdot w_{c_1}^p(x) \cdot w_{c_2}^{p^2}(x) \ \ldots \pmod{p}.$$

Corollary: With the hypotheses of Theorem 3 and with m defined by (1.3),

$$u_m(k;\ x) \equiv u_{c_0}(k;\ x) \cdot w_{c_1}^p(x) \cdot w_{c_2}^{p^2}(x) \ \ldots \pmod{p}.$$

5. GENERAL RESULTS

For each integer $k \geqslant 0$, let $\{F_n(k)\}$ and $\{G_n(k)\}$, $n = 0, 1, 2, \ldots$, be polynomials in an arbitrary number of indeterminates with coefficients that are integral (mod p) for $p > 2k$. We use the notation $F_n(0) = F_n$ and $G_n(0) = G_n$, and we assume $F_0 = G_0 = 1$. For each m of the form (1.3), suppose

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1}^p \cdot F_{c_2}^{p^2} \cdots \pmod{p}, \tag{5.1}$$

$$G_m(k) \equiv G_{c_0}(k) \cdot G_{c_1}^p \cdot G_{c_2}^{p^2} \cdots \pmod{p}. \tag{5.2}$$

For each integer $k \geqslant 0$, define $H_n(k)$ and $Q_n(k)$ by means of

$$\binom{n+k}{n} H_n(k) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n+k}{r} \binom{n+k}{r+k} F_r(k) G_{n-r}(k) \tag{5.3}$$

and

$$\binom{n+k}{k} F_n(k) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n+k}{r} \binom{n+k}{r+k} Q_r(k) G_{n-r}(k). \tag{5.4}$$

Theorem 4: Let the sequences $\{H_n(k)\}$ and $\{Q_n(k)\}$ be defined by (5.3) and (5.4), respectively, and let $H_j = H_j(0)$, $Q_j = Q_j(0)$. If p is a prime, $0 \leqslant s \leqslant p - 2k$, then

$$H_{np+s}(k) \equiv H_s(k) \cdot H_{np} \pmod{p}. \tag{5.5}$$

If $G_0(k) \not\equiv 0 \pmod{p}$, we also have

$$Q_{np+s}(k) \equiv Q_s(k) \cdot Q_{np} \pmod{p}. \tag{5.6}$$

Proof: From (5.3), we have

$$\begin{aligned}
\binom{s+k}{s} H_{np+s}(k) &\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} (-1)^{n+s+r+j} \binom{np}{rp}^2 \binom{s+k}{j} \binom{s+k}{j+k} F_{rp+j}(k) G_{np-rp+s-j}(k) \\
&\equiv \sum_{j=0}^{s} (-1)^{s+j} \binom{s+k}{j} \binom{s+k}{j+k} F_j(k) G_{s-j}(k) \cdot \sum_{r=0}^{n} (-1)^{n+r} \binom{n}{r}^2 F_r^p G_{n-r}^p \\
&\equiv \binom{s+k}{s} H_s(k) \cdot H_n^p \equiv \binom{s+k}{s} H_s(k) \cdot H_{np} \pmod{p}.
\end{aligned}$$

This completes the proof of (5.5).

As for (5.6), we first observe that for $n = 0$ and $0 \leqslant s < p - 2k$, congruence (5.6) is valid. Assume that (5.6) is true for all $rp + j < np + s$ with $0 \leqslant j < p - 2k$. Then, from (5.4), we have

$$\begin{aligned}
\binom{s+k}{s} F_{np+s}(k) &\equiv \sum_{j=0}^{s} \sum_{r=0}^{n} (-1)^{n+s+r+j} \binom{np}{rp}^2 \binom{s+k}{j} \binom{s+k}{j+k} Q_{rp+j}(k) G_{np-rp+s-j}(k) \\
&\equiv \sum_{j=0}^{s} (-1)^{s-j} \binom{s+k}{j} \binom{s+k}{j+k} Q_j(k) G_{s-j}(k) \cdot \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r}^2 Q_r^p G_{n-r}^p \\
&\quad - \binom{s+k}{s} Q_s(k) G_0(k) Q_n^p + \binom{s+k}{s} Q_{np+s}(k) G_0(k) \\
&\equiv \binom{s+k}{s} F_s(k) \cdot F_n^p - \binom{s+k}{s} Q_s(k) G_0(k) Q_n^p \\
&\quad + \binom{s+k}{s} Q_{np+s}(k) G_0(k) \pmod{p}.
\end{aligned}$$

Now, since $F_{np+s}(k) \equiv F_s(k) \cdot F_{np} \pmod{p}$, we have

$$Q_{np+s}(k) \equiv Q_s(k) \cdot Q_n^p \equiv Q_s(k) \cdot Q_{np} \pmod{p},$$

and the proof is complete.

<u>Corollary (Carlitz)</u>: Using the hypotheses of Theorem 4 with m defined by (1.3) and $k = 0$,

$$H_m \equiv H_{c_0} \cdot H_{c_1}^p \cdot H_{c_2}^{p^2} \cdots \pmod{p},$$

$$Q_m \equiv Q_{c_0} \cdot Q_{c_1}^p \cdot Q_{c_2}^{p^2} \cdots \pmod p.$$

Corollary: Using the hypotheses of Theorem 4 with m defined by (1.3),

$$H_m(k) \equiv H_{c_0}(k) \cdot H_{c_1}^p \cdot H_{c_2}^{p^2} \cdots \pmod p.$$

If $G_0(k) \not\equiv 0 \pmod p$, we also have

$$Q_m(k) \equiv Q_{c_0}(k) \cdot Q_{c_1}^p \cdot Q_{c_2}^{p^2} \cdots \pmod p.$$

6. APPLICATIONS

As an application of Theorem 4, for each integer $k \geqslant 0$ consider the expansion

$$(k!)^{r+s-1} \frac{f_k(x_1 z) \cdots f_k(x_r z)}{f_k(y_1 z) \cdots f_k(y_s z)} = \sum_{n=0}^{\infty} F_n(k) \frac{z^m}{n!(n+k)!}, \tag{6.1}$$

where $f_k(z)$ is defined by (1.1), r, s are arbitrary nonnegative integers, and the x_i, y_i are indeterminates (not necessarily distinct). By (1.1) and (3.1), $F_n(k)$ is a polynomial in $x_1, \ldots, x_r$, and $y_1, \ldots, y_s$ with coefficients that are integral (mod p) if $p > 2k$. The following result may be stated.

Theorem 5: If m is of the form (1.3), then the polynonial $F_m(k)$ defined by (6.1) satisfies

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1}^p \cdot F_{c_2}^{p^2} \cdots \pmod p,$$

where $F_j = F_j(0)$. In particular, if the x_i, y_i are replaced by rational numbers that are integral (mod p), then

$$F_m(k) \equiv F_{c_0}(k) \cdot F_{c_1} F_{c_2} \cdots \pmod p.$$

As a special case of (6.1), we may take

$$(k!)^{r-1}\{f_k(z)\}^{-r} = \sum_{n=0}^{\infty} u_n^{(r)}(k) \frac{z^n}{n!(n+k)!}.$$

Then the $u_n^{(r)}(k)$ are integral (mod p) if $p > 2k$, and they satisfy

$$u_m^{(r)}(k) \equiv u_{c_0}^{(r)}(k) \cdot u_{c_1}^{(r)}(0) \cdot u_{c_2}^{(r)}(0) \cdots \pmod p$$

for all r (positive or negative).

7. THE NUMBERS $u_n(1)$

For $n = 0, 1, 2, \ldots$, let $w_n = u_n(0)$ and let $u_n = u_n(1)$. The positive integers w_n were studied by Carlitz [3] and were shown to satisfy (1.4) (with $k = 0$). Since the u_n are also positive integers, it may be of interest to examine their properties in more detail. The generating function and recurrence formula are given by (1.1), (3.1), and (3.2) with $k = 1$. From them we can compute the following values:

$u_0 = u_1 = 1$	$u_5 = 321$
$u_2 = 2$	$u_6 = 3681$
$u_3 = 7$	$u_7 = 56197$
$u_4 = 39$	$u_8 = 1102571$

Suppose that p is an odd prime number and that m is defined by (1.3) with $0 \leqslant c_0 \leqslant p - 3$. Then, by Theorems 1 and 2, we have

$$u_m \equiv u_{c_0} w_{c_1} w_{c_2} \cdots \pmod{p}, \tag{7.1}$$

$$u_{np+(p-1)} \equiv -w_{n+1} \pmod{p}. \tag{7.2}$$

The case $c_0 = p - 2$ is considered in the next theorem. This theorem makes use of the positive integers h_n defined by means of

$$\{J_1(z)\}^2/\{J_0(z)\}^3 = \sum_{n=0}^{\infty} h_n \frac{(z/2)^{2n}}{n!n!} \tag{7.3}$$

These numbers are related to the integers a_n defined by Carlitz [1]:

$$a_n = 2^{2n} n!(n-1)!\sigma_{2n}(0),$$

where $\sigma_{2n}(0)$ is the Rayleigh function. It can be determined from properties of a_n that a generating function is

$$J_1(z)/J_0(z) = \sum_{n=1}^{\infty} a_n \frac{(z/2)^{2n-1}}{n!(n-1)!} \tag{7.4}$$

as well as

$$\{J_1(z)/J_0(z)\}^2 = \sum_{n=1}^{\infty} a_{n+1} \frac{(z/2)^{2n}}{n!n!}. \tag{7.5}$$

Now it follows from (3.1), (7.3), and (7.5) that

$$h_n = \sum_{r=0}^{n-1} \binom{n}{r}^2 w_r a_{n+1-r} \quad (n > 0), \tag{7.6}$$

$$(-1)^n a_{n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}^2 h_r \quad (n > 0). \tag{7.7}$$

The first few values of h_n are $h_0 = 0$, $h_1 = 1$, $h_2 = 8$, $h_3 = 96$, $h_4 = 1720$.

In the proof of Theorem 6, we use the relationship

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r}\binom{n}{r+1} w_{r+1} = (-1)^{n+1} a_{n+1}, \tag{7.8}$$

which follows from (7.4).

Theorem 6: If p is an odd prime number, then

$$u_{np+(p-2)} \equiv u_{p-2} w_n - h_n \pmod{p},$$

where h_n is defined by (7.3).

Proof: The proof is by induction on n. The theorem is true for $n = 0$, since $h_0 = 0$ and $w_0 = 1$. Assume that Theorem 6 is true for $n = 0, \ldots, s - 1$. Then by (3.2), (7.1), (7.2), and (7.8) we have

$$(-1)^{s-1} u_{sp+(p-2)} \equiv \sum_{r=0}^{s} \sum_{j=0}^{p-3} (-1)^{r+j} \binom{sp+p-1}{rp+j}\binom{sp+p-1}{rp+j+1} u_{rp+j}$$
$$+ \sum_{r=0}^{s-1} \sum_{j=p-2}^{p-1} (-1)^{r+j} \binom{sp+p-1}{rp+j}\binom{sp+p-1}{rp+j+1} u_{rp+j}$$

$$\equiv \sum_{r=0}^{s}(-1)^r\binom{s}{r}^2 w_r \cdot \sum_{j=0}^{p-3}(-1)^j\binom{p-1}{j}\binom{p-1}{j+1}u_j + u_{p-2}\sum_{r=0}^{s-1}(-1)^r\binom{s}{r}^2 w_r$$

$$+ \sum_{r=0}^{s-1}(-1)^{r+1}\binom{s}{r}^2 h_r + \sum_{r=0}^{s-1}(-1)^{r+1}\binom{s}{r}\binom{s}{r+1}w_{r+1}$$

$$\equiv (-1)^{s-1}u_{p-2}w_s + (-1)^s h_s + (-1)^{s-1}a_{s+1} + (-1)^s a_{s+1}$$

$$\equiv (-1)^{s-1}(u_{p-2}w_s - h_s) \pmod p.$$

This completes the proof of Theorem 6.

Using (7.7) we can prove, for $p > 2$,

$$h_{np+s} \equiv h_s w_n \pmod p \quad (0 \leqslant s \leqslant p-2),$$

$$h_{np+(p-1)} \equiv h_{p-1}w_n + h_n \pmod p.$$

Theorem 6 can be refined by means of these congruences. For example, if m is defined by (1.3) with $c_0 = p - 2$ and $c_1 = 0$, we have

$$u_m \equiv u_{c_0}w_{c_1}w_{c_2} \; \ldots \pmod p.$$

The proofs in this section are not valid for $p = 2$. However, it is not difficult to show by induction that if $m \not\equiv 2 \pmod 4$ then u_m is odd. The proof is similar to the proofs of Theorems 1-6. If $m \equiv 2 \pmod 4$, we can write

$$m = 4n + 2 = 2^{v+1}j + 2^v - 2$$

for some $v > 1$. Using (3.2) and induction on n, we can prove

$$u_m \equiv \begin{cases} 0 \pmod 2 \text{ if } v \text{ is even,} \\ 1 \pmod 2 \text{ if } v \text{ is odd.} \end{cases}$$

Thus, for $p = 2$, we have the following theorem.

Theorem 7: If $m = c_0 + c_1 2 + c_2 2^2 + \cdots$, with each $c_i = 0$ or 1, then

$$u_m \equiv u_{c_0}u_{c_1}u_{c_2} \; \ldots \pmod 2,$$

unless $m = 2^{v+1}j + 2^v - 2$ with v even, $v \geqslant 2$.

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