SOME PROPERTIES OF BINOMIAL COEFFICIENTS

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1. INTRODUCTION

In 1982, M. Boscarol [1] gave a demonstration of the following property of binomial coefficients:

$$\sum_{i=0}^{m} 2^{-(n+i)}\binom{n+i}{i} + \sum_{j=0}^{n} 2^{-(n+m-j)}\binom{n+m-j}{m} = 2 \tag{1}$$

for each pair of integers $n,\ m \geqslant 0$. For instance, let $m = 4$ and $n = 3$, then we have

$$\sum_{i=0}^{4} 2^{-i-3}\binom{3+i}{i} + \sum_{j=0}^{3} 2^{j-7}\binom{7-j}{4} = 2,$$

i.e.,

$$2^{-3} + 4 \cdot 2^{-4} + 10 \cdot 2^{-5} + 20 \cdot 2^{-6} + 35 \cdot 2^{-7} + 35 \cdot 2^{-7} + 15 \cdot 2^{-6} + 5 \cdot 2^{-5} + 2^{-4} = 2.$$

The purpose of this note is to present a generalization of (1).

2. MAIN RESULTS

Theorem 1: For each pair of integers $n,\ m \geqslant 0$ and $r > 0$, the following identity holds:

$$\sum_{i=0}^{m} r^{m-i}\binom{n+i}{i} = \binom{n+m+1}{m} + (r-1)\sum_{i=0}^{m-1} r^{i}\binom{n+m-i}{n+1}. \tag{2}$$

Proof: For $m = 0$, we have

$$\binom{n}{0} = 1 = \binom{n+1}{0}$$

from the definition. We now show that the formula for $m + 1$ follows from the formula for m.

$$\sum_{i=0}^{m+1} r^{(m+1)-i}\binom{n+i}{i} = \binom{n+m+1}{m+1} + r\sum_{i=0}^{m} r^{m-i}\binom{n+i}{i}$$

$$= \binom{n+m+1}{m+1} + r\Bigg\{\binom{n+m+1}{m}$$

$$+ (r-1)\sum_{i=0}^{m-1} r^{i}\binom{n+m-i}{n+1}\Bigg\}, \text{ by assumption}$$

$$= \left\{\binom{n + m + 1}{m + 1} + \binom{n + m + 1}{m}\right\}$$

$$+ (r - 1)\left\{\binom{n + m + 1}{m} + \sum_{i=0}^{m-1} r^{i+1}\binom{n + m - i}{n + 1}\right\}$$

$$= \binom{n + m + 2}{m + 1} + (r - 1)\sum_{i=-1}^{m-1} r^{i+1}\binom{n + m - i}{n + 1}$$

$$= \binom{n + (m + 1) + 1}{m + 1} + (r - 1)\sum_{j=0}^{m} r^{j}\binom{n + (m + 1) - j}{n + 1},$$

completing our proof.

Theorem 2: For each pair of integers n, $m \geqslant 0$ and $r > 0$, define

$$L(n, m; r) = \sum_{i=0}^{m} r^{m-i}\binom{n + i}{i} + \sum_{j=0}^{n} r^{j}\binom{n + m - j}{m}, \tag{3}$$

then $L(n, m; r)$ satisfies the following recursive form:

$$L(n + 1, m + 1; r) = L(n, m + 1; r) + L(n + 1, m; r)$$

and

$$L(0, n; r) = L(n, 0; r) = \sum_{j=0}^{n} r^{j} + 1.$$

Proof: By (3), we have

$$L(0, n; r) = L(n, 0; r) = \sum_{j=0}^{n} r^{j} + 1.$$

Using a dummy variable, we obtain

$$L(n, m; r) = \sum_{i=0}^{m} r^{m-i}\binom{n + i}{i} + \sum_{j=0}^{n} r^{n-j}\binom{m + j}{j}$$

or (4)

$$L(n, m; r) = \sum_{i=0}^{m} r^{i}\binom{n + m - i}{n} + \sum_{j=0}^{n} r^{j}\binom{n + m - j}{m}.$$

Since

$$\sum_{i=0}^{m+1} r^{i}\binom{n + m + 1 - i}{n} + \sum_{i=0}^{m} r^{i}\binom{n + m + 1 - i}{n + 1}$$

$$= \sum_{i=0}^{m+1} r^{i}\left\{\binom{n + m + 1 - i}{n} + \binom{n + m + 1 - i}{n + 1}\right\} = \sum_{i=0}^{m+1} r^{i}\binom{n + m + 2 - i}{n + 1}$$

and

$$\sum_{j=0}^{n} r^{j}\binom{n + m + 1 - j}{m + 1} + \sum_{j=0}^{n+1} r^{j}\binom{n + m + 1 - j}{m} = \sum_{j=0}^{n+1} r^{j}\binom{n + m + 2 - j}{m + 1},$$

we have

$$L(n + 1, m; r) + L(n, m + 1; r)$$

$$= \left\{\sum_{i=0}^{m} r^{i}\binom{n + m + 1 - i}{n + 1} + \sum_{j=0}^{n+1} r^{j}\binom{n + m + 1 - j}{m}\right\}$$

$$+ \left\{\sum_{i=0}^{m+1} r^{i}\binom{n + m + 1 - i}{n} + \sum_{j=0}^{n} r^{j}\binom{n + m + 1 - j}{m + 1}\right\}$$

$$= \sum_{i=0}^{m-1} r^i \binom{n + m + 2 - i}{n + 1} + \sum_{j=0}^{n+1} r^j \binom{n + m + 2 - j}{m + 1} = L(n + 1, m + 1; r).$$

In fact, the reverse of this theorem is also true by the generating function method.

Theorem 3: For each pair of integers n, $m \geqslant 0$ and $r > 0$, we have

$$L(n, m; r) = \sum_{i=0}^{m} (r - 1)^i \binom{n + m + 1}{m - i} + \sum_{j=0}^{n} (r - 1)^j \binom{n + m + 1}{n - j}. \tag{5}$$

Proof: By (2) and the dummy variable, we have

$$\sum_{i=0}^{m} r^{m-i} \binom{n + i}{i} = \binom{n + m + 1}{m} + (r - 1) \sum_{i=0}^{m-1} r^i \binom{n + m - i}{n + 1}$$

$$= \binom{n + m + 1}{m} + (r - 1) \sum_{j=0}^{m-1} r^{(m-1)-j} \binom{n + 1 + j}{j}.$$

Repeating the above procedure, we obtain

$$\sum_{i=0}^{m} r^{m-i} \binom{n + i}{i} = \sum_{i=0}^{m} (r - 1)^i \binom{n + m + 1}{m - i}, \tag{6}$$

completing our proof.

Corollary 1: For each pair of integers n, $m \geqslant 0$, the following identity holds:

$$\sum_{i=0}^{m} 2^{m-i} \binom{n + i}{i} = \sum_{i=0}^{m} \binom{n + m + 1}{i}. \tag{7}$$

Proof: Taking $r = 2$ in (6), we have

$$\sum_{i=0}^{m} 2^{m-i} \binom{n + i}{i} = \sum_{i=0}^{m} \binom{n + m + 1}{m - i} = \sum_{j=0}^{m} \binom{n + m + 1}{j}, \text{ by } j = m - i.$$

Corollary 2: For each pair of integers n, $m \geqslant 0$, we have

$$L(n, m; 2) = 2^{n+m+1}. \tag{8}$$

Proof: $L(n, m; 2) = \sum_{i=0}^{m} 2^{m-i} \binom{n + i}{i} + \sum_{j=0}^{m} 2^j \binom{n + m - j}{m}$

$$= \sum_{i=0}^{m} 2^{m-i} \binom{n + i}{i} + \sum_{j=0}^{n} 2^{n-j} \binom{m + j}{j}$$

$$= \sum_{i=0}^{m} \binom{n + m + 1}{i} + \sum_{j=0}^{n} \binom{n + m + 1}{j}$$

$$= \sum_{i=0}^{m} \binom{n + m + 1}{i} + \sum_{k=m+1}^{n+m+1} \binom{n + m + 1}{k} = 2^{n+m+1}.$$

Dividing identity (8) by 2^{n+m}, we obtain (1).

3. EXAMPLES

Example 1: Take $r = 3$. We have the values of $L(n, m; 3)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	5	14	41	122	365	1094	3281
1	5	10	24	65	187	552	1646	4927
2	14	24	48	113	300	852	2498	7425
3	41	65	113	226	526	1378	3876	11301
4	122	187	300	526	1052	2430	6306	17607
5	365	552	852	1378	2430	4860	11166	28773
6	1094	1646	2498	3876	6306	11166	22332	51105
7	3281	4927	7425	11301	17607	28773	51105	102210

Example 2: Take $r = 4$. We obtain the values of $L(n, m; 4)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	6	22	86	342	1366	5462	21846
1	6	12	34	120	462	1828	7290	29136
2	22	34	68	188	650	2478	9768	38904
3	86	120	188	376	1026	3504	13272	52176
4	342	462	650	1026	2052	5556	18828	71004
5	1366	1828	2478	3504	5556	11112	29940	100944
6	5462	7290	9768	13272	18828	29940	59880	160824
7	21846	29136	38904	52176	71004	100944	160824	321648

Example 3: Take $r = 5$. We have the values of $L(n, m; 5)$ as follows:

$m \backslash n$	0	1	2	3	4	5	6	7
0	2	7	32	157	782	3907	19532	97657
1	7	14	46	203	985	4892	24424	122081
2	32	46	92	295	1280	6172	30596	152677
3	157	203	295	590	1870	8042	38638	191315
4	782	985	1280	1870	3740	11782	50420	241735
5	3907	4892	6172	8024	11782	23564	73984	315719
6	19532	24424	30596	38638	50420	73984	147968	463687
7	97657	122081	152677	191315	241735	315719	463687	927374

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REFERENCE

1. M. Boscarol. "A Property of Binomial Coefficients." *The Fibonacci Quarterly* *20*, no. 3 (1982):249-51.

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