FUNCTIONS OF NON-UNITARY DIVISORS

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1. INTRODUCTION

A divisor d of n is a *unitary divisor* if gcd $(d, n/d) = 1$; in such a case, we write $d \| n$. There is a considerable body of results on functions of unitary divisors (see [2]-[7]). Let $\tau^*(n)$ and $\sigma^*(n)$ denote, respectively, the number and sum of the unitary divisors of n.

We say that a divisor d of n is a *non-unitary divisor* if $(d, n/d) > 1$. If d is a non-unitary divisor of n, we write $d |^{\#} n$. In this paper, we examine some functions of non-unitary divisors.

We will find it convenient to write

$$n = \overline{n} \cdot n^{\#},$$

where $\overline{n}$ is the largest squarefree unitary divisor of n. We call $\overline{n}$ the *squarefree part* of n and $n^{\#}$ the *powerful part* of n. Then, if p is prime, $p | \overline{n}$ implies $p \| n$, while $p | n^{\#}$ implies $p^2 | n$. Naturally, either $\overline{n}$ or $n^{\#}$ can be 1 if required (if n is powerful or squarefree, respectively).

2. THE SUM OF NON-UNITARY DIVISORS FUNCTION

Let $\sigma^{\#}(n)$ be the sum of the non-unitary divisors of n:

$$\sigma^{\#}(n) = \sum_{d |^{\#} n} d.$$

Now, every divisor is either unitary or non-unitary. Because $\overline{n}$ and $n^{\#}$ are relatively prime and the σ and σ^* functions are multiplicative, we have

$$\sigma^{\#}(n) = \sigma(n) - \sigma^*(n) = \sigma(\overline{n})\sigma(n^{\#}) - \sigma^*(\overline{n})\sigma^*(n^{\#}).$$

But $\sigma(\overline{n}) = \sigma^*(\overline{n})$, so

$$\sigma^{\#}(n) = \sigma(\overline{n})\{\sigma(n^{\#}) - \sigma^*(n^{\#})\}.$$

Therefore,

$$\sigma^{\#}(n) = \left\{\prod_{p \| n} (p + 1)\right\} \cdot \left\{\prod_{\substack{p^e \| n \\ e > 1}} \frac{p^{e+1} - 1}{p - 1} - \prod_{\substack{p^e \| n \\ e > 1}} (p^e + 1)\right\}.$$

Note that $\sigma^{\#}(n) = 0$ if and only if n is squarefree, and that $\sigma^{\#}$ is *not* multiplicative.

Recall that an integer n is perfect [unitary perfect] if it equals the sum of its proper divisors [unitary divisors]. This is usually stated as $\sigma(n) = 2n$ [$\sigma^*(n) = 2n$] in order to be dealing with multiplicative functions. But all non-unitary divisors are proper divisors, so the analogous definition here is that n is *non-unitary perfect* if $\sigma^{\#}(n) = n$.

Theorem 1: If $2^p - 1$ is prime, so that $2^{p-1}(2^p - 1)$ is an even perfect number, then $2^{p+1}(2^p - 1)$ is non-unitary perfect.

Proof: Suppose $n = 2^{p+1}(2^p - 1)$, where p is prime. Then

$$\begin{aligned}\sigma^{\#}(n) &= \sigma(2^p - 1)\{\sigma(2^{p+1}) - \sigma^*(2^{p+1})\}\\ &= 2^p[(2^{p+2} - 1) - (2^{p+2} + 1)]\\ &= 2^p(2^{p+1} - 2) = 2^{p+1}(2^p - 1) = n.\end{aligned}$$

A computer search written under our direction by Abdul-Nasser El-Kassar found no other non-unitary perfect numbers less than one million. Accordingly, we venture the following:

Conjecture 1: An integer is non-unitary perfect if and only if it is 4 times an even perfect number.

If $n^{\#}$ is known or assumed, it is relatively easy to search for $\overline{n}$ to see if n is non-unitary perfect. Many cases are eliminated because of having $\sigma^{\#}(n^{\#}) > n^{\#}$. In most other cases, the search fails because $\overline{n}$ would have to contain a repeated factor. For example, if $n^{\#} = 2^2 5^2$, then no $\overline{n}$ will work, for

$$\sigma^{\#}(2^2 5^2) = 7 \cdot 31 - 5 \cdot 26 = 87 = 3 \cdot 29,$$

so $3 \cdot 29 | \overline{n}$; but $2^2 5^2 29 \| n$ implies $3^2 | n$, so $3 | \overline{n}$ is impossible.

The second author generated by computer all powerful numbers not exceeding 2^{15}. Examination of the various cases verified that there is no non-unitary perfect number n with $n^{\#} \leqslant 2^{15}$ except when n satisfies Theorem 1 [i.e., $n = 2^{p+1}(2^p - 1)$, where $2^p - 1$ is prime].

More generally, we say that n is *k-fold non-unitary perfect* if $\sigma^{\#}(n) = kn$, where $k \geqslant 1$ is an integer. We examined all $n^{\#} \leqslant 2^{15}$ and all $n \leqslant 10^6$ and found the k-fold non-unitary perfect numbers $(k > 1)$ listed in Table 1. Based on the profusion of examples and the relative ease of finding them, we hazard the following (admittedly shaky) guess:

Conjecture 2: There are infinitely many k-fold non-unitary perfect numbers.

Table 1. k-fold Non-Unitary Perfect Numbers $(k > 1)$

k	n
2	$2^3 3^2 5 \cdot 7 = 2520$
2	$2^3 3^3 5 \cdot 29 = 31\ 320$
2	$2^3 3^4 5 \cdot 359 = 1\ 163\ 160$
2	$2^7 3^5 71 = 2\ 208\ 384$
2	$2^4 3^2 7 \cdot 13 \cdot 233 = 3\ 053\ 232$
2	$2^7 3^3 31 \cdot 61 = 6\ 535\ 296$
2	$2^5 3^2 7 \cdot 41 \cdot 163 = 13\ 472\ 928$
2	$2^5 5^2 3 \cdot 19 \cdot 37 \cdot 73 = 123\ 165\ 600$
2	$2^7 3^4 47 \cdot 751 = 365\ 959\ 296$
2	$2^4 3^4 11 \cdot 131 \cdot 2357 = 4\ 401\ 782\ 352$
2	$2^{10} 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 5\ 517\ 818\ 880$
3	$2^7 3^2 5^2 \cdot 7 \cdot 13 \cdot 71 = 186\ 076\ 800$
3	$2^8 3^4 5 \cdot 7 \cdot 11 \cdot 53 \cdot 769 = 325\ 377\ 803\ 520$
3	$2^6 3^2 7^2 5 \cdot 13 \cdot 19 \cdot 113 \cdot 677 = 2\ 666\ 567\ 816\ 640$

We say that n is *non-unitary subperfect* if $\sigma^{\#}(n)$ is a proper divisor of n. Because $\sigma^{\#}(18) = 9$ and $\sigma^{\#}(p^2) = p$ if p is prime, we have the following:

Theorem 2: If $n = 18$ or $n = p^2$, where p is prime, then n is non-unitary subperfect.

An examination of all $n^{\#} \leqslant 2^{15}$ and all $n \leqslant 10^6$ found no other non-unitary subperfect numbers, so we are willing to risk the following:

Conjecture 3: An integer n is non-unitary subperfect if and only if $n = 18$ or $n = p^2$, where p is prime.

It is possible to define non-unitary harmonic numbers by requiring that the harmonic mean of the non-unitary divisors be integral. If $\tau^{\#}(n) = \tau(n) - \tau^*(n)$ counts the number of non-unitary divisors, the requirement is that $n\tau^{\#}(n)/\sigma^{\#}(n)$ be integral. We found several dozen examples less than 10^6, including all k-fold non-unitary perfect numbers, as well as numbers of the forms

$$2 \cdot 3p^2,\ p^2(2p - 1),\ 2 \cdot 3p^2(2p - 1),\ 2^{p+1}3(2^p - 1),\ 2^{p+1}3 \cdot 5(2^p - 1),$$
$$\text{and } 2^{p+1}(2p - 1)(2^p - 1),$$

where p, $2p - 1$, and $2^p - 1$ are distinct primes. Many other examples seemed to fit no general pattern.

Recall that integers n and m are amicable [unitary amicable] if each is the sum of the proper divisors [unitary divisors] of the other. Similarly, we say that n and m are *non-unitary amicable* if

$$\sigma^{\#}(n) = m \quad \text{and} \quad \sigma^{\#}(m) = n.$$

Theorem 3: If $2^p - 1$ and $2^q - 1$ are prime, then $2^{p+1}(2^q - 1)$ and $2^{q+1}(2^p - 1)$ are non-unitary amicable.

Proof: Trivial verification.

Thus, there are at least as many non-unitary amicable pairs as there are pairs of Mersenne primes. Our computer search for $n < m$ and $n \leqslant 10^6$ revealed only four non-unitary amicable pairs that are not characterized by Theorem 3:

$$n = 252 = 2^2 3^2 7 \qquad m = 328 = 2^3 41$$
$$n = 3240 = 2^3 3^4 5 \qquad m = 6462 = 2 \cdot 3^2 359$$
$$n = 11616 = 2^5 3 \cdot 11^2 \qquad m = 17412 = 2^2 \cdot 3 \cdot 1451$$
$$n = 11808 = 2^5 3^2 41 \qquad m = 20538 = 2 \cdot 3^2 \cdot 7 \cdot 163$$

3. THE NON-UNITARY ANALOG OF EULER'S FUNCTION

Euler's function

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \prod_{p^e \| n} (p^e - p^{e-1})$$

is usually defined as the number of positive integers not exceeding n that are relatively prime to n. The unitary analog is

$$\varphi^*(n) = n \prod_{p^e \| n} \left(1 - \frac{1}{p^e}\right) = \prod_{p^e \| n} (p^e - 1).$$

Our first task here is to give equivalent alternative definitions for φ and φ^* which will suggest a non-unitary analog. In particular, we may define $\varphi(n)$ as the number of positive integers not exceeding n that are not divisible by any of the divisors $d > 1$ of n. Similarly, $\varphi^*(n)$ may be defined as the number of positive integers not exceeding n that are not divisible by any of the unitary divisors $d > 1$ of n.

Recalling that 1 is never a non-unitary divisor of n, it is natural in light of the alternative definitions of φ and φ^* to define $\varphi^{\#}(n)$ as the number of positive integers not exceeding n that are not divisible by any of the non-unitary divisors of n. By imitating the usual proofs for φ and φ^*, it is easy to show that $\varphi^{\#}$ is multiplicative, and that

$$\varphi^{\#}(n) = \bar{n}\varphi(n^{\#}). \tag{1}$$

The following result neatly connects divisors, unitary divisors, and non-unitary divisors in a, perhaps, unexpected way:

Theorem 4: $\displaystyle\sum_{d|n} \varphi^{\#}(d) = \sigma^*(n).$

Proof: The Dirichlet convolution preserves multiplicativity, and $\varphi^{\#}$ is multiplicative, so we need only check the assertion for prime powers. In light of (1), doing so is easy, because the sum telescopes:

$$\begin{aligned}\sum_{d|p^e} \varphi^{\#}(d) &= \varphi^{\#}(1) + \varphi^{\#}(p) + \varphi^{\#}(p^2) + \cdots + \varphi^{\#}(p^e)\\ &= 1 + p + (p^2 - p) + \cdots + (p^e - p^{e-1})\\ &= 1 + p^e = \sigma^*(p^e).\end{aligned}$$

It is well known that

$$\sum_{d|n} \varphi(d) = n \quad \text{and} \quad \sum_{d\|n} \varphi^*(d) = n,$$

and one might anticipate a similar result involving $\varphi^{\#}$. However, the situation is a bit complicated. We write

$$\sum_{d|^{\#}n} \varphi^{\#}(d) = \sum_{d|n} \varphi^{\#}(d) - \sum_{d\|n} \varphi^{\#}(d). \tag{2}$$

Now, both convolutions on the right side of (2) preserve multiplicativity and, as a result, it is possible to obtain the following:

Theorem 5: $\displaystyle\sum_{d|^{\#}n} \varphi^{\#}(d) = \sigma(\bar{n})\left\{\sigma^*(n^{\#}) - \prod_{p^e\|n^{\#}}(p^e - p^{e-1} + 1)\right\}$

Theorem 5 was first obtained by Scott Beslin in his Master's thesis [1], written under the direction of the first author of this paper.

Two questions arise in connection with Theorem 5. First, is it possible to find a subset $S(n)$ of the divisors of n for which

$$\sum_{d\in S(n)} \varphi^{\#}(d) = n?$$

It is indeed possible to do so. Let $\omega(n)$ be the number of distinct primes that divide n. We say that d is an *ω-divisor* of n if $d|n$ and $\omega(d) = \omega(n)$, i.e., if every prime that divides n also divides d. Let $\Omega(n)$ denote the set of all ω-divisors of n.

Theorem 6: $\sum_{d \in \Omega(n)} \varphi^{\#}(d) = n.$

Proof: Trivial if $\omega(n) = 0$. But if $\omega(n) = 1$, the sum is that in the proof of Theorem 4 except that the term "$\varphi^{\#}(1) = 1$" is missing. Easy induction on $\omega(n)$, using the multiplicativity of $\varphi^{\#}$, completes the proof.

The other question that arises from Theorem 5 is whether it is possible to have

$$\sum_{d|n} \varphi^{\#}(d) = n, \quad n > 1. \tag{3}$$

We know of ten solutions to (3), and they are given in Table 2. By Theorem 5, if n satisfies (3), then

$$\sigma(\overline{n})/\overline{n} = n^{\#}/\left\{\sigma^*(n^{\#}) - \prod_{p^e \| n^{\#}} (p^e - p^{e-1} + 1)\right\}. \tag{4}$$

This observation makes it easy to search for $\overline{n}$ if $n^{\#}$ is known. The first eight numbers in Table 2 are the only solutions to (3) with $1 < n \leqslant 2^{15}$.

Table 2. Solutions to (3), Ordered by $n^{\#}$

n	$n^{\#}$	$\overline{n}$
5 220	$2^2 3^2$	$5 \cdot 29$
3 960	$2^3 3^2$	$5 \cdot 11$
8 447 040	$2^6 3^2$	$5 \cdot 7 \cdot 419$
6 773 440	$2^7 3^2$	$5 \cdot 7 \cdot 167$
18 685 336 320	$2^8 3^2$	$5 \cdot 7 \cdot 139 \cdot 1667$
341 863 562 880	$2^7 3^3$	$5 \cdot 7 \cdot 29 \cdot 41 \cdot 2377$
1 873 080	$2^3 3^2 11^2$	$5 \cdot 43$
1 018 887 932 160	$2^8 3^4$	$5 \cdot 7 \cdot 19 \cdot 37 \cdot 1997$
20 993 596 382 889 043 200	$2^8 3^2 5^2$	$7 \cdot 19 \cdot 2393 \cdot 23929 \cdot 47857$
357 174 165 248	$2^{13} 3^2$	$7 \cdot 11 \cdot 13 \cdot 47 \cdot 103$

It seems unlikely that one could completely characterize the solutions to (3). However, we do know the following:

Theorem 7: If $n > 1$ satisties (3), then $n^{\#}$ is divisible by at least two distinct primes.

Proof: We must have $n^{\#} > 1$ because $\sigma(\overline{n}) \geqslant \overline{n}$ with equality only if $\overline{n} = 1$. Suppose $n^{\#} = p^e$, where p is prime and $e \geqslant 2$. Then, from (4), we have $\sigma(\overline{n})/\overline{n} = p$. If $p = 2$, then $\overline{n}$ is an odd squarefree perfect number, which is impossible. Now, $\overline{n}$ is squarefree, and any odd prime that divides $\overline{n}$ contributes at least one factor 2 to $\sigma(\overline{n})$, and since $p \neq 2$, we have $2 \| \overline{n}$. Then $\overline{n} = 2q$, where q is prime, and the requirement $\sigma(\overline{n})/\overline{n} = p$ forces $q = 3/(2p - 3)$, which is impossible if $p > 2$.

We strongly suspect the following is true:

Conjecture 4: If n satisfies (3), then $n^{\#}$ is even.

If the right side of (4) does not reduce, then Conjecture 4 is true: If we suppose that $n^{\#}$ is odd, then $4 | \sigma^*(n^{\#})$, as $n^{\#}$ has at least two distinct prime divisors by Theorem 7. Then, it is easy to see that the denominator of the right side of (4) is of the form $4k - 1$, and if the right side of (4) does not reduce, then $\overline{n}$ is of the form $4k - 1$, whence $4 | \sigma(\overline{n})$, making (4) impossible. Thus, any counterexample to Conjecture 4 requires that the fraction on the right side of (4) reduce.

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