ANALOGS OF SMITH'S DETERMINANT*

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Over a century ago, according to Dickson [1], H. J. S. Smith [3] showed that

$$\begin{vmatrix} (1, 1) & \dots & (1, j) & \dots & (1, n) \\ \vdots & & \vdots & & \vdots \\ (i, 1) & \dots & (i, j) & \dots & (i, n) \\ \vdots & & \vdots & & \vdots \\ (n, 1) & \dots & (n, j) & \dots & (n, n) \end{vmatrix} = \varphi(1)\ \varphi(2)\ \dots\ \varphi(n),$$

where (i, j) is the greatest common divisor of i and j, and φ is Euler's function. P. Mansion [2] proved a generalization of Smith's result: If

$$f(m) = \sum_{d|m} g(d),$$

and we write $f(i, j)$ for $f(\gcd(i, j))$, then

$$\begin{vmatrix} f(1, 1) & \dots & f(1, j) & \dots & f(1, n) \\ \vdots & & \vdots & & \vdots \\ f(i, 1) & & f(i, j) & & f(i, n) \\ \vdots & & \vdots & & \vdots \\ f(n, i) & \dots & f(n, j) & \dots & f(n, n) \end{vmatrix} = g(1)\ g(2)\ \dots\ g(n).$$

Note that Mansion's result becomes Smith's when $f(m) = m$, because

$$m = \sum_{d|m} \varphi(d).$$

In this paper, we present an extension of Mansion's result to a wide class of arithmetic convolutions.

Suppose $S(m)$ defines some set of divisors of m for each m. If $d|m$, we say that d is an S-divisor of m if $d \in S(m)$. We will denote by $(i, j)_S$ the largest common S-divisor of i and j.

Now m might or might not be an element of $S(m)$, as can be seen if we let $S(m)$ be the largest squarefree divisor of m. Also, the property

$$d \in S(i) \cap S(j) \text{ if and only if } d \in S((i, j)_S)$$

might or might not be true. It is true if $S(m)$ consists of all the divisors of

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m, but not if $S(m)$ consists of all divisors d of m for which $(d, m/d) > 1$, for then 6 is the largest common S-divisor of 12 and 24, and 2 is an S-divisor of 12 and 24, but not of 6.

We come now to the promised generalization:

Theorem: Let $S(m)$ and $(i, j)_S$ be defined as above. If

(1) $m \in S(m)$ for each m,

(2) $d \in S(i) \cap S(j)$ if and only if $d \in S((i, j)_S)$, and

(3) $f(m) = \sum\limits_{d \in S(m)} g(d)$,

then

$$\begin{vmatrix} f((1, 1)_S) & \ldots & f((1, j)_S) & \ldots & f((1, n)_S) \\ \vdots & & \vdots & & \vdots \\ f((i, 1)_S) & \ldots & f((i, j)_S) & \ldots & f((i, n)_S) \\ \vdots & & \vdots & & \vdots \\ f((n, 1)_S) & \ldots & f((n, j)_S) & \ldots & f((n, n)_S) \end{vmatrix} = g(1) \ldots g(n).$$

Proof: Assume the hypotheses, and define

$$S(a, b) = \begin{cases} 1 & \text{if } b \in S(a), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $S(a, b) = 0$ if $b > a$, and by (1) we have $S(a, a) = 1$ for each a. Now, $S(i, d)S(j, d)$ is 0 unless d is an S-divisor of both i and j, in which case the product is 1, and by (2) and (3) it is easy to see that

$$\begin{aligned} f((i, j)_S) = {} & S(i, 1)S(j, 1)g(1) + S(i, 2)S(j, 2)g(2) \\ & + \cdots + S(i, n)S(j, n)g(n) \end{aligned}$$

for each i and j. Then

$$[f((i, j)_S)] = A \cdot B,$$

where

$$A = \begin{bmatrix} S(1, 1) & S(1, 2) & \ldots & S(1, i) & \ldots & S(1, n) \\ S(2, 1) & S(2, 2) & \ldots & S(2, i) & \ldots & S(2, n) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(i, 1) & S(i, 2) & \ldots & S(i, i) & \ldots & S(i, n) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(n, 1) & S(n, 2) & \ldots & S(n, i) & \ldots & S(n, n) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ S(2, 1) & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ S(i, 1) & S(i, 2) & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ S(n, 1) & S(n, 2) & \ldots & S(n, i) & \ldots & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} S(1, 1)g(1) & S(2, 1)g(1) & \dots & S(j, 1)g(1) & \dots & S(n, 1)g(1) \\ S(1, 2)g(2) & S(2, 2)g(2) & \dots & S(j, 2)g(2) & \dots & S(n, 2)g(2) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(1, j)g(j) & S(2, j)g(j) & \dots & S(j, j)g(j) & \dots & S(n, j)g(j) \\ \vdots & \vdots & & \vdots & & \vdots \\ S(1, n)g(n) & S(2, n)g(n) & \dots & S(j, n)g(n) & \dots & S(n, n)g(n) \end{bmatrix}$$

$$= \begin{bmatrix} g(1) & S(2, 1)g(1) & \dots & S(j, 1)g(1) & \dots & S(n, 1)g(1) \\ 0 & g(2) & \dots & S(j, 2)g(2) & \dots & S(n, 2)g(2) \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & g(j) & \dots & S(n, j)g(j) \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & g(n) \end{bmatrix}$$

The theorem then follows from the observations

$$\det A = 1 \quad \text{and} \quad \det B = g(1)\ g(2)\ \dots\ g(n). \quad \blacksquare$$

In particular, if $S(m)$ consists of all divisors of m, the theorem yields Mansion's result. Another special case of some interest arises if we let $S(m)$ consist of the unitary divisors of m: We say that d is a unitary divisor of m if gcd $(d, m/d) = 1$. Let $(i, j)^*$ be the largest common unitary divisor of i and j. Also, let $\tau^*(m)$ and $\sigma^*(m)$ be the number and sum, respectively, of the unitary divisors of m. Then $g(d) = 1$ and $g(d) = d$, respectively, yield

$$|\tau^*((i, j)^*)| = 1 \quad \text{and} \quad |\sigma^*((i, j)^*)| = n!$$

REFERENCES

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