# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
RAYMOND E. WHITNEY

*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

H-418 *Proposed by Lawrence Somer, Washington, D.C.*

Let $m > 1$ be a positive integer. Suppose that $m$ itself is a general period of the Fibonacci sequence modulo $m$; that is,

$$F_{n+m} \equiv F_n \pmod{m}$$

for all nonnegative integers $n$. Show that $24 | m$.

H-419 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $P_0$, $P_1$, ... be the sequence of Pell numbers defined by

$$P_0 = 0,\ P_1 = 1, \text{ and } P_n = 2P_{n-1} + P_{n-2} \text{ for } n \in \{2, 3, \ldots\}.$$

Show that

(a) $9\sum_{k=0}^{n} kF_kP_k = 3(n+1)(F_nP_{n+1} + F_{n+1}P_n) - F_{n+2}P_{n+2} - F_nP_n + 2,$

(b) $9\sum_{k=0}^{n} kL_kP_k = 3(n+1)(L_nP_{n+1} + L_{n+1}P_n) - L_{n+2}P_{n+2} - L_nP_n,$

(c) $F_{m+n+2}P_{n+2} + F_{m+n}P_n \equiv 3(n+1)F_m + L_m \pmod{9}.$

(d) $L_{m+n+2}P_{n+2} + L_{m+n}P_n \equiv 3(n+1)L_m + 5F_m \pmod{9},$

where $n$ is a nonnegative integer and $m$ any integer.

H-420 *Proposed by Peter Kiss, Teachers Training College, Eger, Hungary and Andreas N. Philippou, University of Patras, Patras, Greece*

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2^{n-1}}}{2^{2^n} - 1} = 1.$

## SOLUTIONS

Editorial Note: H-307 was listed as an unsolved problem. However, the solution for H-307 was inadvertently placed in the same issue as the problem.

### Return from the Dead

H-211 *Proposed by S. Krishman, Orissa, India*
*(Vol. 11, no. 1, February 1973)*

A. Show that $\binom{2n}{n}$ is of the form $2n^3k + 2$ when $n$ is prime and $n > 3$.

B. Show that $\binom{2n-2}{n-1}$ is of the form $n^3k - 2n^2 - n$ when $n$ is prime.

$\binom{m}{j}$ represents the binomial coefficient, $m!/(j!(m-j)!)$.

*Solution by Paul S. Bruckman, Fair Oaks, CA*

Consider the expansion:

(1) $$(x+1)(x+2)\cdots(x+n-1) = x^{n-1} + A\,x^{n-2} + \cdots + A_{n-2} + A_{n-1},$$

where $A_k$ is the sum of the products of the $k$ different members of the set $1, 2, \ldots, n-1$.

If $n \geqslant 3$ is prime, Theorem 113 in [1] states:

(2) $$A_k \equiv 0 \pmod{n},\ k = 1, 2, \ldots, n-2.$$

Moreover, Wolstenholme's Theorem (Theorem 115 in [1]) states:

(3) $$A_{n-2} \equiv 0 \pmod{n^2}, \text{ provided } n > 3.$$

Also, Wilson's Theorem states:

(4) $$A_{n-1} = (n-1)! \equiv -1 \pmod{n}.$$

If $n > 3$ (and prime), then, by (4):

$$(n!)^2 = n^2(an-1)^2 \text{ for some integer } a.$$

Also, setting $x = n$ in (1) and applying (2), (3), and (4), we obtain

$$(2n-1)!/n! = an - 1 + n \cdot bn^2 + n^2 \cdot cn + dn^3 = an - 1 + fn^3$$

(for integers $b$, $c$, $d$, and $f$; here "$a$" is the same integer as in the previous statement). Therefore,

$$\binom{2n}{n} = \frac{2n}{n!}\,\frac{(2n-1)!}{n!} = \frac{2}{an-1}(an - 1 + fn^3)$$
$$= 2 + \frac{2fn^3}{an-1} \equiv 2 \pmod{2n^3};$$

this proves part (A) of the problem.

Now,

$$\binom{2n-2}{n-1} = \frac{n}{2(2n-1)}\binom{2n}{n};$$

hence, again, if $n$ is a prime greater than 3,

$$\binom{2n-2}{n-1} = \frac{n(2+2kn^3)}{2(2n-1)} \quad \text{(for some integer } k\text{), so}$$

$$\binom{2n-2}{n-1} = \frac{n(kn^3+1)}{2n-1} \equiv (kn^3+1)(-n)(1+2n) \pmod{n^3}$$

$$= -n - 2n^2 \pmod{n^3};$$

this proves part (B).

## Reference

1. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed. (Oxford: Clarendon Press, 1960), pp. 86-88.

## Another Ancient One

H-213 *Proposed by V. E. Hoggatt, Jr., San Jose State University, (deceased) (Vol. 11, no. 1, February 1973)*

A. Let $A_n$ be the left adjusted Pascal triangle, with $n$ rows and columns and 0's above the main diagonal. Thus,

$$A_n = \begin{bmatrix} 1 & 0 & & & \dots & 0 \\ 1 & 1 & 0 & & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n \times n}$$

Find $A_n \cdot A_n^T$, where $A_n^T$ represents the transpose of matrix $A_n$.

B. Let

$$C_n = \begin{bmatrix} 1 & 0 & 0 & & & \dots & 0 \\ 0 & 1 & 0 & & & \dots & 0 \\ 0 & 1 & 1 & 0 & & \dots & 0 \\ 0 & 0 & 2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n \times n}$$

where the $i^{\text{th}}$ column of the matrix $C_n$ is the $i^{\text{th}}$ row of Pascal's triangle adjusted to the main diagonal and the other entries are zeros. Find $C_n \cdot A_n^T$.

*Solution by Paul S. Bruckman, Fair Oaks, CA*

Part (A): We see that

$$a_{ij} = \binom{i}{j}, \quad 0 \leqslant i,\ j \leqslant n-1,$$

with the convention that $a_{ij} = 0$ outside this range. Hence, if $B = AA^T$, and

$m = \min(i, j)$, then

$$b_{ij} = \sum_{k=0}^{m} a_{ik}a_{jk} = \sum_{k=0}^{m} \binom{i}{k}\binom{j}{k} = \sum_{k=0}^{m} \binom{i}{k}\binom{j}{j-k} = \binom{i+j}{j}$$

(using Vandermonde's convolution), provided $0 \leqslant i, j \leqslant n - 1$.

This is a symmetric matrix, whose rows (and columns) are the coefficients of powers of $(1 - x)^{-1}$.

Part (B): We see that

$$c_{ij} = \binom{j}{i-j}, \text{ where } 0 \leqslant j \leqslant i \leqslant 2j \leqslant 2n - 2, \; c_{ij} = 0 \text{ elsewhere.}$$

If $D = CA^T$, and if $u = [\frac{1}{2}(i + 1)]$, then

$$d_{ij} = \sum_{k=u}^{m} c_{ik}a_{jk} = \sum_{k=u}^{m} \binom{k}{i-k}\binom{j}{k}.$$

Note that

$$F(x) = \sum_{i=0}^{\infty} d_{ij}x^i = \sum_{i=0}^{m} x^i \sum_{k=u}^{m} \binom{k}{i-k}\binom{j}{k} = \sum_{k=0}^{j} \binom{j}{k} \sum_{i=k}^{2k} \binom{k}{i-k} x^i$$

$$= \sum_{k=0}^{j} \binom{j}{k} x^k \sum_{i=0}^{k} \binom{k}{i} x^i = \sum_{k=0}^{j} \binom{j}{k} x^k (1 + x)^k$$

$$= (1 + x + x^2)^j = \sum \binom{n_1 + n_2 + n_3}{n_1, n_2, n_3} x^i,$$

where the last sum is over nonnegative integers $n_1$, $n_2$, $n_3$, such that

$$n_1 + n_2 + n_3 = j, \; n_1 + 2n_2 + 3n_3 = i + j.$$

Thus,

$$d_{ij} = \sum \binom{n_1 + n_2 + n_3}{n_1, n_2, n_3},$$

over the range indicated, for $0 \leqslant i, j \leqslant n - 1$ ($d_{ij} = 0$ if $i > 2j$). Hence, the columns of $D = CA^T$ are the rows of the Pascal *trinomial* triangle truncated after $n$ terms.

## Some Operator

H-397 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
*(Vol. 24, no. 2, May 1986)*

For any positive integer $n$, define the function $F_n$ on $\mathbf{C}$ as follows:

$$F_n(x) \equiv (g^n - 1)(x), \tag{1}$$

where $g$ is the operator

$$g(x) \equiv x^2 - 2. \tag{2}$$

[Thus, $F_3(x) = \{(x^2 - 2)^2 - 2\}^2 - 2 - x = x^8 - 8x^6 + 20x^4 - 16x^2 - x + 2$.] Find all $2^n$ zeros of $F_n$.

*Solution by the proposer*

We find that the following substitution yields fruitful results:

$$x = 2\cos\theta. \tag{3}$$

For then

$$g(x) = 4\cos^2\theta - 2 = 2\cos 2\theta,\ g^2(x) = 2\cos(2^2\theta),\ \text{etc.},$$

$$g^n(x) = 2^n\cos(2^n\theta).$$

Hence, $F_n(x) = 2\cos(2^n\theta) - 2\cos\theta$. Setting $F_n(x) = 0$ yields:

$$2^n\theta = \pm\theta + 2k\pi \text{ for all integers } k;$$

since

$$\theta = 2k\pi/(2^n \pm 1),$$

we may restrict $k$ to the values $0, 1, \ldots, (2^n \pm 1) - 1$.

We consider the two cases implied by the ± sign above separately. If $\theta = 2k\pi/(2^n - 1)$, we may further restrict $k$ to the values $0, 1, \ldots, 2^{n-1} - 1$; for if $2^{n-1} \leqslant k \leqslant 2^n - 2$, then $k' \equiv 2^n - 1 - k$ satisfies $1 \leqslant k' \leqslant 2^{n-1} - 1$, i.e., $k'$ repeats the same values previously assumed by $k$, except for zero. Moreover,

$$\cos(2k'\pi/(2^n - 1) = \cos(2\pi - 2k\pi/(2^n - 1)) = \cos(2k\pi/(2^n - 1));$$

thus, $k \in [0, 2^{n-1} - 1]$ generates all zeros of $F_n$ under this case.

If $\theta = 2k\pi/(2^n + 1)$, we may restrict $k$ to the values $1, 2, \ldots, 2^{n-1}$; for if $2^{n-1} + 1 \leqslant k \leqslant 2^n$, then $k' \equiv 2^n + 1 - k$ satisfies $1 \leqslant k' \leqslant 2^{n-1}$, i.e., $k'$ repeats the same values previously assumed by $k$. Moreover, as before,

$$\cos(2k'\pi/(2^n + 1)) = \cos(2k\pi/(2^n + 1)).$$

Thus, all zeros of $F_n$ are generated in this case by the values $k \in [1, 2^{n-1}]$.

The zeros of $F_n$ found above are $2^n$ in number, which is expected in an equation of degree $2^n$. Further, they are distinct, since $(2^n - 1)$ and $(2^n + 1)$ are relatively prime, and all zeros in each of the two cases considered above are distinct. Thus, the zeros of $F_n$ are as follows:

$$2\cos\{2(k - 1)\pi/(2^n - 1)\}$$

or

$$2\cos\{2k\pi/(2^n + 1)\},\ k = 1, 2, \ldots, 2^{n-1}.$$

For example,

$$F_3(x) = \prod_{k=1}^{4}\{x - 2\cos(2(k - 1)\pi/7)\}\{x - 2\cos(2k\pi/9)\}.$$

## A Piece of Pie

H-398 *Proposed by Ambati Jaya Krishna, Freshman, Johns Hopkins University (Vol. 24, no. 2, May 1986)*

Let

$$a + b + c + d + e = \left(\sum_1^\infty\left(\frac{(-1)^{n+1}}{2n - 1}\,\frac{2}{3}\cdot 9^{1-n} + 7^{1-2n}\right)\right)^2$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = \frac{45}{512}\sum_1^\infty n^{-4},$$

$a, b, c, d, e \in \mathbb{R}$. What are the values of $a$, $b$, $c$, $d$, and $e$ if $e$ is to attain its maximum value?

*Solution by Paul S. Bruckman, Fair Oaks, CA*

$$\left(\sum_0^\infty \frac{(-1)^n}{2n+1}\left(2\left(\frac{1}{3}\right)^{2n+1} + \left(\frac{1}{7}\right)^{2n+1}\right)\right)^2 = \left(2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2$$

$$= \left(\tan^{-1}\left(\frac{2/3}{1 - 1/9}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2 = \left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)^2$$

$$= \left(\tan^{-1}\left(\frac{3/4 + 1/7}{1 - 3/28}\right)\right)^2 = (\tan^{-1} 1)^2 = (\pi/4)^2,$$

or

$$a + b + c + d + e = \pi^2/16. \tag{1}$$

Also,

$$\sum_1^\infty n^{-4} = \xi(4) = \pi^4/90,$$

so

$$a^2 + b^2 + c^2 + d^2 + e^2 = \pi^4/1024 = (\pi^2/32)^2. \tag{2}$$

To simplify the computations, we make the following substitutions:

$$a = \pi^2/32x_1,\ b = \pi^2/32x_2,\ \ldots,\ e = \pi^2/32x_5. \tag{3}$$

We observe that $e$ is maximized iff $x_5$ is. Then, the equivalents of (1) and (2) are:

$$S = \sum_1^5 x_k = 2; \tag{4}$$

$$Q = \sum_1^5 x_k^2 = 1. \tag{5}$$

This is an extremal problem with constraints. Such problems may be solved by using Lagrange's method of multipliers (see Angus E. Taylor, *Advanced Calculus* [Ginn & Co., 1955], pp. 198-204). We form the function

$$u = u(x_1, x_2, x_3, x_4, x_5; \lambda_1, \lambda_2) = x_5 + \lambda_1 S + \lambda_2 Q, \tag{6}$$

where $\lambda_1$ and $\lambda_2$ are indeterminate "Multipliers." According to Lagrange's method, *all* extremal values of $x_5$, subject to the constraints given by (4) and (5), are provided as solutions of the equations

$$\frac{\partial u}{\partial x_k} = 0,\ k = 1, 2, 3, 4, 5, \text{ together with (4) and (5).} \tag{7}$$

We then obtain:

$$\lambda_1 + 2x_k\lambda_2 = 0,\ k = 1, 2, 3, 4; \tag{8}$$

$$1 + \lambda_1 + 2x_5\lambda_2 = 0. \tag{9}$$

We observe that we cannot have $\lambda_2 = 0$; for, if $\lambda_2 = 0$, then (8) implies $\lambda_1 = 0$. But then (9) would imply $1 = 0$, clearly impossible. Since $\lambda_2 \neq 0$, it follows from (8) that for any extremal solutions of the problem, we must have $x_1 = x_2 = x_3 = x_4$. Let $x$ denote the common value of the $x_k$'s ($k = 1, 2, 3, 4$), $y$ the corresponding extremal value(s) of $x_5$. We then obtain, from (4) and (5):

$$4x + y = 2; \tag{10}$$

$$4x^2 + y^2 = 1. \tag{11}$$

We may readily solve (1) and (11), obtaining the two solutions

$$(x, y) = (\tfrac{1}{2}, 0), \text{ or } (3/10, 4/5). \tag{12}$$

Since this provides *all* extremal values $y$, we see that $x_5$ is maximized at $y = 4/5$ iff $x = 3/10$. Returning to our original notation [i.e., using (3)], it follows that $e$ assumes its maximum value of $\pi^2/40$ iff

$$a = b = c = d = 3\pi^2/320.$$

*Also solved by C. Georghiou, L. Kuipers, J.-Z. Lee & J.-S. Lee, and the proposer.*

Rules, Rules, Rules

H-399 *Proposed by M. Wachtel, Zurich, Switzerland*
*(Vol. 24, no. 2, May 1986)*

The twin sequences: $$\frac{L_{1+6n} - 1}{2} = 0, 14, 260, 4674, 83880, \ldots$$

and $$\frac{L_{5+6n} - 1}{2} = 5, 99, 1785, 32039, \ldots$$

are representable by infinitely many identities, partitioned into several groups of similar structure (see *The Fibonacci Quarterly* *24*, no. 2 [May 1986], p. 186 for details). Find the construction rules for $S_n$ for each group.

*Solution by Paul S. Bruckman, Fair Oaks, CA*

The Group I formulas for $S_m$ are as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} - 1) = \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}\{(\alpha^{3m-1} - \alpha^{-(3m-1)})(\alpha^{6n-9m+6+2k} + \alpha^{-(6n-9m+6+2k)}) - 1\}, \tag{1}$$

where $k = -1$ or $+1$, $m = 1, 2, 3, \ldots$ .

Depending on whether $m$ is odd or even, these may be expressed as follows:

$$\begin{aligned}\tfrac{1}{2}(L_{6n+3+2k} - 1) &= \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}(5F_{3m-1}F_{6n-9m+6+2k} - 1), \ m \text{ odd}; &\text{(1a)}\\ &= \tfrac{1}{2}(L_{6m-2} - 1)L_{6n-6m+5+2k} + \tfrac{1}{2}(L_{3m-1}L_{6n-9m+6+2k} - 1), \ m \text{ even}.\end{aligned}$$

The corresponding Group II and III formulas are as follows, with a similar dichotomy as indicated below:

$$\tfrac{1}{2}(L_{6n+3+2k} - 1) = \tfrac{1}{2}(L_{6m-4} - 1)L_{6n-6m+7+2k} + \tfrac{1}{2}\{(\alpha^{3m-2} - \alpha^{-(3m-2)})(\alpha^{6n-9m+9+2k} + \alpha^{-(6n-9m+9+2k)}) - 1\}; \tag{2}$$

$$= \tfrac{1}{2}(L_{6m-4} - 1)L_{6n-6m+7+2k} + \tfrac{1}{2}(L_{3m-2}L_{6n-9m+9+2k} - 1),\ m \text{ odd}; \quad (2a)$$
$$= \quad '' \quad + \tfrac{1}{2}(5F_{3m-2}F_{6n-9m+9+2k} - 1),\ m \text{ even};$$

$$= \tfrac{1}{2}(L_{6m-4} + 1)L_{6n-6m+7+2k} \quad (3)$$
$$- \tfrac{1}{2}\{(\alpha^{3m-2} + \alpha^{-(3m-2)})(\alpha^{6n-9m+9+2k} - \alpha^{-(6n-9m+9+2k)}) + 1\};$$

$$= \tfrac{1}{2}(L_{6m-4} + 1)L_{6n-6m+7+2k} - \tfrac{1}{2}(5F_{3m-2}F_{6n-9m+9+2k} + 1),\ m \text{ odd}; \quad (3a)$$
$$= \quad '' \quad - \tfrac{1}{2}(L_{3m-2}L_{6n-9m+9+2k} + 1),\ m \text{ even}.$$

**Proof of (1):** The right member of (1) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+7+2k} - L_{6n-6m+5+2k} + L_{6n-6m+5+2k} - L_{6n-12m+7+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

**Proof of (2):** The right member of (2) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+11+2k} - L_{6n-6m+7+2k} + L_{6n-6m+7+2k} - L_{6n-12m+11+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

**Proof of (3):** The right member of (3) simplifies as follows:

$$\tfrac{1}{2}(L_{6n+3+2k} + L_{6n-12m+11+2k} + L_{6n-6m+7+2k} - L_{6n-6m+7+2k} - L_{6n-12m+11+2k} - 1)$$
$$= \tfrac{1}{2}(L_{6n+3+2k} - 1). \quad \text{Q.E.D.}$$

*Also solved by J.-Z. Lee & J.-S. Lee as well as the proposer.*

**Editorial Note:** Might as well dedicate this issue to Paul S. Bruckman.

◆◇◆◇◆