A NEW GENERALIZATION OF DAVISON'S THEOREM

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(Submitted June 1986)

1. INTRODUCTION

In [3] Davison proved that

$$\sum_{n\geq 1} \frac{1}{2^{\lfloor n\alpha \rfloor}} = \cfrac{1}{2^{F_0} + \cfrac{1}{2^{F_1} + \cfrac{1}{2^{F_2} + \cdots}}}, \text{ with } \alpha = \frac{1+\sqrt{5}}{2},$$

where $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, and $\lfloor x \rfloor$ is the greatest integer $\leq x$. In [1] the authors found the simple continued fraction for

$$T(x, C) = (C - 1) \sum_{n\geq 1} \frac{1}{C^{\lfloor nx \rfloor}}, \text{ with real } x > 1 \text{ and } C > 1.$$

In this paper, we shall prove a new generalization of Davison's Theorem (see Theorem 1).

2. CONVENTIONS AND USEFUL THEOREMS

Throughout this paper, make the following conventions:

$$\alpha = \frac{1+\sqrt{5}}{2}.$$

Let F_n be defined for negative n by $F_{n+2} = F_{n+1} + F_n$.

Define Y_n by: Y_0 and Y_1 are given real numbers such that $Y_0 + Y_1\alpha > 0$, and all other values of Y_n are defined by $Y_{n+2} = Y_{n+1} + Y_n$, n any integer.

Also, throughout, let the Fibonacci representation of an integer $K \geq 1$ be written as

$$K = F_{V_1} + F_{V_2} + \cdots + F_{V_n}, \tag{1}$$

where $2 \leq V_1 <_2 V_2 <_2 \ldots <_2 V$ and $a <_2 b$ means that $a + 2 \leq b$.

Define the function $e(K)$, for K an integer ≥ 0, by

$$e(K) = 0 \text{ if } K = 0;$$

otherwise,

$$e(K) = F_{V_1 - 1} + F_{V_2 - 1} + \cdots + F_{V_n - 1}, \text{ where } K \text{ has the representation (1).}$$

In the paper [4], setting $\alpha = \frac{\sqrt{5}+1}{2}$ gives

$$e(k) = \lfloor (k+1)\alpha^{-1} \rfloor, \text{ for } k \geq 0. \tag{2}$$

The convergence ranges for the series in this paper can easily be justified by comparing the series to geometric series. Because of the limit passing below, the convergence ranges for the continued fractions are also justified.

From [6], we will use the Euler-Minding Theorem:

If $\frac{A_p}{B_p} = 1 + \frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+\cdots}\,\frac{C_p}{1}$, where $\{C_k\}$ is a sequence of nonzero real numbers for $k \geq 1$, then,

$$A_p = 1 + \sum_{n \geq 1,\ 1 \leq V_1 <_2 \cdots <_2 V_n \leq P} C_{V_1} C_{V_2} \cdots C_{V_n},$$

and

$$B_p = 1 + \sum_{n \geq 1,\ 2 \leq V_1 <_2 \cdots <_2 V_n \leq P} C_{V_1} C_{V_2} \cdots C_{V_n}.$$

Actually, all that is needed is the following corollary:

Write $A\ (C_1, C_2, \ldots, C_p) = A_p$, then notice $B_p = A_{p-1}(C_2, C_3, \ldots, C_p)$. Now, let $P \to \infty$ and we have:

$$1 + \frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+\cdots} = \frac{A_\infty(C_1, C_2, \ldots)}{A_\infty(C_2, C_3, \ldots)}. \tag{3}$$

Notice that the indices on the summation for A_∞ will be:

$$n \geq 1,\ 1 \leq V_1 <_2 V_2 <_2 \cdots <_2 V_n.$$

3. THE MAIN THEOREMS

Theorem 1:

$$\frac{\displaystyle\sum_{n\geq 1}\left(\frac{1}{C}\right)^{Y_0 n + Y_{-1}\lfloor n\alpha^{-1}\rfloor}}{\displaystyle\sum_{n\geq 1}\left(\frac{1}{C}\right)^{Y_1 n + Y_0\lfloor n\alpha^{-1}\rfloor}} = C^{Y_{-1}} + \frac{1}{C^{Y_0}+}\,\frac{1}{C^{Y_1}+}\,\frac{1}{C^{Y_2}+}\,\frac{1}{C^{Y_3}+\cdots},$$

where $C > 1$.

Proof: Set $C_n = a^{F_{n-1}} b^{F_n}$ in (3), with $|a|, |b| \leq 1$, not both 1, to get

$$1 + \frac{a^{F_0}b^{F_1}}{1+}\,\frac{a^{F_1}b^{F_2}}{1+\cdots} = \frac{1 + \displaystyle\sum_{n\geq 1,\ 1\leq V_1 <_2 \cdots <_2 V_n} a^{F_{v_1-1}+\cdots+F_{v_n-1}} b^{F_{v_1}+\cdots+F_{v_n}}}{1 + \displaystyle\sum_{n\geq 1,\ 1\leq V_1 <_2 \cdots <_2 V_n} a^{F_{v_1}+\cdots+F_{v_n}} b^{F_{v_1+1}+\cdots+F_{v_n+1}}}$$

Denote the numerator by $F(a, b)$ and the denominator by $G(a, b)$.

Now,

$$F(b, ab) = 1 + \sum_{n\geq 1,\ 1\leq V_1 <_2 \cdots <_2 V_n} a^{F_{v_1}+\cdots+F_{v_n}} b^{F_{v_1+1}+\cdots+F_{v_n+1}} = G(a, b). \tag{4}$$

Hence, we have

$$\frac{F(a,\ b)}{F(b,\ ab)} = 1 + \frac{a^{F_0}b^{F_1}}{1+}\ \frac{a^{F_1}b^{F_2}}{1+\cdots}. \tag{5}$$

From this, it follows that

$$\frac{F(b,\ ab)}{F(ab,\ ab^2)} = 1 + \frac{a^{F_1}b^{F_2}}{1+}\ \frac{a^{F_2}b^{F_3}}{1+\cdots},$$

so we find that

$$F(a,\ b) = F(b,\ ab) + bF(ab,\ ab^2), \tag{6}$$

with $|a|,\ |b| \leqslant 1$, and not both 1.

An expansion for $F(a,\ b)$ could now be reached by setting

$$F(a,\ b) = \sum k_{n,m}a^nb^m,\text{ with } n,\ m \geqslant 0,$$

and equating coefficients in (6), but this route is tedious. Instead, notice that if in (4) the exponent of b is k, then the exponent of a will be $e(k)$ and because of Zeckendorf's Theorem (see [2]), k will range over the integers > 0. Hence,

$$F(b,\ ab) = 1 + \sum_{n\geqslant 1} a^{e(n)}b^n = \sum_{n\geqslant 0} a^{e(n)}b^n.$$

Thus, we also get

$$F(a,\ b) = \sum_{n\geqslant 0} a^{n-e(n)}b^{e(n)}.$$

Using (2), we have

$$F(a,\ b) = \sum_{n\geqslant 0} a^{n-\lfloor (n+1)\alpha^{-1}\rfloor}b^{\lfloor (n+1)\alpha^{-1}\rfloor}, \tag{7}$$

and

$$F(b,\ ab) = \sum_{n\geqslant 0} a^{\lfloor (n+1)\alpha^{-1}\rfloor}b^n. \tag{8}$$

Let $a = C^A$ and $b = C^B$ in (7) and (8) to get

$$F(C^A,\ C^B) = \sum_{n\geqslant 1} C^{A(n-1)+(B-A)\lfloor n\alpha^{-1}\rfloor}, \tag{9}$$

and

$$F(C^B,\ C^{A+B}) = \sum_{n\geqslant 1} C^{B(n-1)+A\lfloor n\alpha^{-1}\rfloor}. \tag{10}$$

Set $A = Y_0 - Y_1$ and $B = -Y_0$ in (10) to get

$$F\left(\left(\frac{1}{C}\right)^{Y_0},\ \left(\frac{1}{C}\right)^{Y_1}\right) = \sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{Y_0(n-1)+(Y_1-Y_0)\lfloor n\alpha^{-1}\rfloor},\quad |C| > 1, \tag{11}$$

or set $A = -Y_0$ and $B = -Y_1$ in (10) to get

$$F\left(\left(\frac{1}{C}\right)^{Y_1},\ \left(\frac{1}{C}\right)^{Y_0+Y_1}\right) = \sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{Y_1(n-1)+Y_0\lfloor n\alpha^{-1}\rfloor},\quad |C| > 1. \tag{12}$$

From (5), we see that

$$\frac{F(C^{Y_0},\ C^{Y_1})}{F(C^{Y_1},\ C^{Y_0+Y_1})} = 1 + \frac{C^{Y_0F_0+Y_1F_1}}{1\ +}\ \frac{C^{Y_0F_1+Y_1F_2}}{1\ +}\ \frac{C^{Y_0F_2+Y_1F_3}}{1\ +\ \cdots},\ 0 < C < 1.$$

It is easy to show by induction that $Y_n = Y_0F_{n-1} + Y_1F_n$, for integer n; hence,

$$\frac{F(C^{Y_0},\ C^{Y_1})}{F(C^{Y_1},\ C^{Y_0+Y_1})} = 1 + \frac{C^{Y_1}}{1\ +}\ \frac{C^{Y_2}}{1\ +}\ \frac{C^{Y_3}}{1\ +\ \cdots},\ 0 < C < 1.$$

Replacing C with its reciproval variable,

$$\frac{F\left(\left(\frac{1}{C}\right)^{Y_0},\ \left(\frac{1}{C}\right)^{Y_1}\right)}{F\left(\left(\frac{1}{C}\right)^{Y_1},\ \left(\frac{1}{C}\right)^{Y_0+Y_1}\right)} = 1 + \frac{C^{-Y_1}}{1\ +}\ \frac{C^{-Y_2}}{1\ +}\ \frac{C^{-Y_3}}{1\ +\ \cdots},\ C > 1$$

$$= 1 + \frac{C^{Y_0}C^{-Y_1}}{C^{Y_0}\ +}\ \frac{C^{Y_0}C^{Y_1}C^{-Y_2}}{C^{Y_1}\ +}\ \frac{C^{Y_1}C^{Y_2}C^{-Y_3}}{C^{Y_2}\ +}\ \frac{C^{Y_2}C^{Y_3}C^{-Y_4}}{C^{Y_3}\ +\ \cdots},$$

$$C > 1,$$

(by the equivalence relation (3.1) of [7])

$$= 1 + \frac{C^{Y_0-Y_1}}{C^{Y_0}\ +}\ \frac{1}{C^{Y_1}\ +}\ \frac{1}{C^{Y_2}\ +}\ \frac{1}{C^{Y_3}\ +\ \cdots},\ C > 1.$$

Hence,

$$\frac{F\left(\left(\frac{1}{C}\right)^{Y_0},\ \left(\frac{1}{C}\right)^{Y_1}\right)C^{-Y_0}}{F\left(\left(\frac{1}{C}\right)^{Y_1},\ \left(\frac{1}{C}\right)^{Y_0+Y_1}\right)C^{-Y_1}} = C^{Y_{-1}} + \frac{1}{C^{Y_0}\ +}\ \frac{1}{C^{Y_1}\ +}\ \frac{1}{C^{Y_2}\ +}\ \frac{1}{C^{Y_3}\ +\ \cdots},\ C > 1.$$

Substituting in (11) and (12) and simplifying yields the theorem.

Theorem 2: $\displaystyle\sum_{n\geqslant 1} C^{A(n-1)+(B-A)\lfloor n\alpha^{-1}\rfloor} = \sum_{n\geqslant 1} C^{B(n-1)+A\lfloor n\alpha^{-1}\rfloor} + C^{A(n-1)+B\lfloor n\alpha\rfloor}$, for $|C| < 1$.

Proof: Let $a = C^{A+B}$ and $b = C^{A+2B}$ in (7) and simplify to get

$$F(C^{A+B},\ C^{A+2B}) = \sum_{n\geqslant 1} C^{(A+B)(n-1)+B\lfloor n\alpha^{-1}\rfloor}. \tag{13}$$

Let $a = C^A$ and $b = C^B$ in (6) to get

$$F(C^A,\ C^B) = F(C^B,\ C^{A+B}) + C^BF(C^{A+B},\ C^{A+2B}). \tag{14}$$

Now substitute (9), (10), and (13) into (14) and simplify to get the theorem.

Corollary 1: If $T = \displaystyle\sum_{n\geqslant 1} C^{Y_kn+Y_{k+1}\lfloor n\alpha\rfloor}$, for $C < 1$, then $T_{k+2} = T_k - C^{-Y_{k+1}}\,T_{k+1}$, where k is any integer.

Proof: Let $A = Y_{k+2}$ and $B = Y_{k+3}$ in Theorem 2 and simplify.

Corollary 2: $\sum_{n\geqslant 1} C^{F_k n + F_{k+1}\lfloor n\alpha\rfloor}$, for $C < 1$, can be evaluated in terms of $\sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor}$ and rational functions of C for any integer k. For example,

$$\sum_{n\geqslant 1} C^{n+2\lfloor n\alpha\rfloor} = (1 + C^{-1})\sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor} - (1 + C)^{-1}. \tag{15}$$

Proof: Put $Y_k = F_k$ in Corollary 1. Notice that

$$T_{-1} = \sum_{n\geqslant 1} C^n = \frac{C}{C - 1} \quad \text{and} \quad T_0 = \sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor}.$$

Now Corollary 2 follows by induction using Corollary 1. For example, we find

$$T_1 = \frac{C}{C - 1} - T_0 \quad \text{or} \quad \sum_{n\geqslant 1} C^{\lfloor n\alpha^2\rfloor} = \frac{C}{C - 1} - \sum_{n\geqslant 1} C^{\lfloor n\alpha\rfloor},$$

which is easily verified by Beatty's Theorem (see [5]). Applying Corollary 1 another time gives (15).

Corollary 3: $\sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{F_k n + F_{k+1}\lfloor n\alpha\rfloor}$ is trancendental for integer $k \neq -1$ and integer $C > 1$.

Proof: From Corollary 2 we can see that the sum for $k \neq -1$ and rational function of C added to a rational function of C multiplied by $\sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{\lfloor n\alpha\rfloor}$ which is transcendental by setting $\alpha = (1 + \sqrt{5})/2$ in [1]. We can show by induction that the rational function which multiplies $\sum_{n\geqslant 1}\left(\frac{1}{C}\right)^{\lfloor n\alpha\rfloor}$ is nonzero; hence, the corollary follows.

Corollary 4: If A and B are integers not both zero, then the number of times that any integer occurs in the sequence

$$A(n - 1) + (B - A)\lfloor n\alpha^{-1}\rfloor, \text{ for } n \geqslant 1,$$

is equal to the total number of times that integer occurs in the following sequences:

$$B(n - 1) + A\lfloor n\alpha^{-1}\rfloor, \text{ for } n \geqslant 1, \text{ and } A(n - 1) + B\lfloor n\alpha\rfloor, \text{ for } n \geqslant 1.$$

Proof: The proof follows immediately from Theorem 2.

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