A NOTE ON THE GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

This note is an extension of the results of L. Carlitz [1] concerning the problem of the multiple generating functions of F_k and L_k, where F_k and L_k are the k^{th} Fibonacci and Lucas numbers, respectively. Our proofs are very similar to those given by Carlitz [1]. Notation and content of [3] are assumed, when required.

Consider the sequence of numbers W_n defined by the second-order recurrence relation

$$W_{n+2} = pW_{n+1} - qW_n, \text{ with } W_0 = a \text{ and } W_1 = b, \tag{1}$$

i.e.,

$$W_n = W_n(a, b; p, q),$$

where a, b, p, and q are real numbers, usually integers.

From [2] and [3], we have

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where

$$\begin{cases} \alpha = (p + d)/2, \ \beta = (p - d)/2, \ d = (p^2 - 4q)^{1/2}, \\ A = (b - a\beta)/d, \ B = (a\alpha - b)/d. \end{cases} \tag{3}$$

Standard methods enable us to derive the following generating function for $\{W_n\}$,

$$\sum_{n=0}^{\infty} W_n x^n = \{a + (b - ap)x\}/(1 - px + qx^2). \tag{4}$$

2. MAIN RESULTS

Define

$$\begin{cases} C_W(x_1, \ldots, x_m) = \prod_{j=1}^{m} (1 - qx_j)(1 - V_2 x_j + q^2 x_j^2), \\ W_1(x_1, \ldots, x_m; k) = A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)(1 - \beta^2 x_j) \\ \qquad\qquad + B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)(1 - \alpha^2 x_j), \\ W_2(x_1, \ldots, x_m; k) = \alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)(1 - \beta^2 x_j) \\ \qquad\qquad + \beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)(1 - \alpha^2 x_j), \end{cases}$$

where $V_n = W_n(2, p; p, q)$. That is, $V_0 = 2$, $V_1 = p$, $V_2 = p^2 - 2q$, ...

Theorem 1: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1 + \cdots + n_m + k} W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}$$

$$= W_1(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Proof: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1 + \cdots + n_m + k} W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}$$

$$= \sum_{n_1, \ldots, n_m = 0}^{\infty} (A\alpha^{n_1 + \cdots + n_m + k} + B\beta^{n_1 + \cdots + n_m + k}) W_{n_1} \ldots W_{n_m} x_1^{n_1} \ldots x_m^{n_m}, \text{ by (2)}$$

$$= A\alpha^k \sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1} \ldots W_{n_m} (\alpha x_1)^{n_1} \ldots (\alpha x_m)^{n_m}$$

$$+ B\beta^k \sum_{n_1, \ldots, n_m = 0}^{\infty} W_{n_1} \ldots W_{n_m} (\beta x_1)^{n_1} \ldots (\beta x_m)^{n_m}$$

$$= A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)/(1 - p\alpha x_j + q\alpha^2 x_j^2)$$

$$+ B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)/(1 - p\beta x_j + q\beta^2 x_j^2), \text{ by (4)}$$

$$= A\alpha^k \cdot \prod_{j=1}^{m} (a + (b - ap)\alpha x_j)/\{(1 - \alpha^2 x_j)(1 - \alpha\beta x_j)\}$$

$$+ B\beta^k \cdot \prod_{j=1}^{m} (a + (b - ap)\beta x_j)/\{(1 - \beta^2 x_j)(1 - \alpha\beta x_j)\}, \text{ by (3)}$$

$$= W_1(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Using a method similar to that used for Theorem 1, we have

Theorem 2: $$\sum_{n_1, \ldots, n_m = 0}^{\infty} V_{n_1 + \cdots + n_m + k} W_{n_1} \cdots W_{n_m} x_1^{n_1} \cdots x_m^{n_m}$$

$$= W_2(x_1, \ldots, x_m; k)/C_W(x_1, \ldots, x_m).$$

Taking $m = 2$ in Theorems 1 and 2, we obtain

Corollary 1: $$\sum_{m, n = 0}^{\infty} W_{m+n+k} W_m W_n x^m y^n = W_1(x, y; k)/C_W(x, y)$$

and $$\sum_{m, n = 0}^{\infty} V_{m+n+k} W_m W_n x^m y^n = W_2(x, y; k)/C_W(x, y),$$

where

$$C_W(x, y) = (1 - qx)(1 - qy)(1 - V_2 x + q^2 x^2)(1 - V_2 y + q^2 y^2),$$

$$\begin{aligned} W_1(x, y; k) = {} & a^2 W_k + a((b - ap)W_{k+1} - aq^2 W_{k-2})(x + y) \\ & - a(b - ap)q^2 W_{k-1}(x + y)^2 + ((b - ap)^2 W_{k+2} + a^2 q^4 W_{k-4})xy \\ & + (b - ap)(aq^4 W_{k-3} - (b - ap)q^2 W_k)xy(x + y) \\ & + (b - ap)^2 q^4 W_{k-2} x^2 y^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y; k) = {} & a^2 V_k + a((b - ap)V_{k+1} - aq^2 V_{k-2})(x + y) \\ & - a(b - ap)q^2 V_{k-1}(x + y)^2 + ((b - ap)^2 V_{k+2} + a^2 q^4 V_{k-4})xy \\ & + (b - ap)(aq^4 V_{k-3} - (b - ap)q^2 V_k)xy(x + y) \\ & + (b - ap)^2 q^4 V_{k-2} x^2 y^2. \end{aligned}$$

Taking $k = 0$ in Corollary 1, we derive

Corollary 2: $$\sum_{m, n = 0}^{\infty} W_{m+n} W_m W_n x^m y^n = W_1(x, y; 0)/C_W(x, y)$$

and $$\sum_{m, n = 0}^{\infty} V_{m+n} W_m W_n x^m y^n = W_2(x, y; 0)/C_W(x, y),$$

where

$$\begin{aligned} W_1(x, y; 0) = {} & a^3 + a(b^2 - a^2(p^2 - q))(x + y) + a(b - ap)^2 q(x + y)^2 \\ & + ((b - ap)^2(bp - aq) + a^3(p^4 - 3p^2 q + q^2) \\ & - a^2 b(p^3 - 2pq))xy + aq(b - ap)(ap(p^2 - q) \\ & - b(p^2 - 2q))xy(x + y) + (b - ap)^2 q^2 (a(p^2 - q) - bp)x^2 y^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y; 0) = {} & 2a^2 + a(bp - 2a(p^2 - q))(x + y) - a(b - ap)pq(x + y)^2 \\ & + ((b - ap)^2(p^2 - 2q) + a^2(p^4 - 4p^2 q + 2q^2))xy \\ & + q(b - ap)(a(p^3 - 3pq) - 2q(b - ap))xy(x + y) \\ & + (b - ap)^2 q^2 (p^2 - 2q)x^2 y^2. \end{aligned}$$

Obviously, all formulas of §2 in [1] are special cases of Theorems 1 and 2 and Corollaries 1 and 2 since $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$. Note that (2.2), (2.3), and (2.8) of [1] are misprinted.

Taking $m = 3$ and $k = 0$ in Theorems 1 and 2, we have

Corollary 3: $$\sum_{m,n,k=0}^{\infty} W_{m+n+k} W_m W_n W_k x^m y^n z^k = W_1(x, y, z; 0)/C_W(x, y, z)$$

and $$\sum_{m,n,k=0}^{\infty} V_{m+n+k} W_m W_n W_k x^m y^n z^k = W_2(x, y, z; 0)/C_W(x, y, z),$$

where

$$\begin{aligned} C_W(x, y, z) = {} & (1 - qx)(1 - qy)(1 - qz)(1 - V_2x + q^2x^2)(1 - V_2y \\ & + q^2y^2)(1 - V_2z + q^2z^2), \end{aligned}$$

$$\begin{aligned} W_1(x, y, z; 0) = {} & a^4 + a((b - ap)W_1 - aq^2W_{-2})(x + y + z) + a((b - ap)^2W_2 \\ & + a^2q^4W_{-4})(xy + yz + zx) + ((b - ap)^3W_3 - a^3q^6W_{-6})xyz \\ & - a^2q^2(b - ap)W_{-1}(x + y + z)^2 + a^2q(b - ap) \\ & \cdot (W_3 - (b - ap)q)(x + y + z)(xy + yz + zx) \\ & + a(b - ap)^2q^4W_{-2}(xy + yz + zx)^2 - q^2(b - ap) \\ & \cdot ((b - ap)^2W_1 + a^2q^4W_{-5})xyz(x + y + z) + (b - ap)^2 \\ & \cdot ((b - ap)W_{-1} + aq^2W_{-4})q^4xyz(xy + yz + zx) \\ & - (b - ap)^3q^6W_{-3}x^2y^2z^2, \end{aligned}$$

$$\begin{aligned} W_2(x, y, z; 0) = {} & 2a^3 + a((b - ap)p - aV_2)(x + y + z) + a((b - ap)^2V_2 \\ & + a^2V_4)(xy + yz + zx) + ((b - ap)^3V_3 - a^3V_6)\ xyz \\ & - a^2(b - ap)pq(x + y + z)^2 + a^2q(b - ap) \\ & \cdot (V_3 - (b - ap)q)(x + y + z)(xy + yz + zx) \\ & + aq^2(b - ap)^2V_2(xy + yz + zx)^2 - q(b - ap) \\ & \cdot ((b - ap)^2pq + a^2V_5)xyz(x + y + z) + q^2(b - ap)^2 \\ & \cdot ((b - ap)pq + aV_4)xyz(xy + yz + zx) \\ & - (b - ap)^3q^3V_3x^2y^2z^2. \end{aligned}$$

Obviously, all formulas of §3 in [1] are also special cases of Theorems 1 and 2 and Corollary 3. Note that (3.2)-(3.5) of [1] are misprinted.

Define

$$W(k, m) = A^k\alpha^m - (-B)^k\beta^m.$$

From (2), (3), and the binomial theorem, we have

Lemma 1: $$d^{k-1}W(k, m) = \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}W_{m-r}.$$

Proof:
$$\begin{aligned}
d^{k-1}W(k, m) &= d^{k-1}(A^k\alpha^m - (-B)^k\beta^m), \text{ by } (2)\\
&= A\alpha^m(dA)^{k-1} + B\beta^m(-dB)^{k-1}\\
&= A\alpha^m(b - a\beta)^{k-1} + B\beta^m(b - a\alpha)^{k-1}, \text{ by } (3)\\
&= A\alpha^m\sum_{r=0}^{k-1}\binom{k-1}{r}(-a\beta)^r b^{k-r-1} + B\beta^m\sum_{r=0}^{k-1}\binom{k-1}{r}(-a\alpha)^r b^{k-r-1}\\
&= \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}(A\alpha^{m-r} + B\beta^{m-r}), \text{ by } (3)\\
&= \sum_{r=0}^{k-1}\binom{k-1}{r}(-aq)^r b^{k-r-1}W_{m-r}, \text{ by } (2).
\end{aligned}$$

Define
$$\begin{cases}
D_W(x, y, z) = d^2(1 - V_2x + q^2x^2)(1 - V_2y + q^2y^2)(1 - V_2z + q^2z^2)\\
W_3(x, y, z; k) = \displaystyle\sum_{j=0}^{3}(-q^2)^j h_j\left\{\sum_{r=0}^{2}\binom{2}{r}(-aq)^r b^{2-r}W_{3k-2j-r}\right\},
\end{cases}$$

where h_j is the j^{th} elementary symmetric function of x, y, and z. That is to say, $h_0 = 1$, $h_1 = x + y + z$, $h_2 = xy + yz + zx$, and $h_3 = xyz$.

Theorem 3:
$$\sum_{m,n,t=0}^{\infty} W_{m+n+k}W_{n+t+k}W_{t+m+k}x^m y^n z^t = W_3(x, y, z; k)/D_W(x, y, z)$$
$$+ eq^k d^{-2}\Sigma(W_k - q^2W_{k-2}x)/\{(1 - V_2x + q^2x^2)(1 - qy)(1 - qz)\}.$$

Proof:
$$\sum_{m,n,t=0}^{\infty} W_{m+n+k}W_{n+t+k}W_{t+m+k}x^m y^n z^t$$
$$\begin{aligned}
&= \sum_{m,n,t=0}^{\infty}(A\alpha^{m+n+k} + B\beta^{m+n+k})(A\alpha^{n+t+k} + B\beta^{n+t+k})\\
&\qquad\cdot(A\alpha^{t+m+k} + B\beta^{t+m+k})x^m y^n z^t, \text{ by } (2)\\
&= A^3\alpha^{3k}/\{(1 - \alpha^2x)(1 - \alpha^2y)(1 - \alpha^2z) + \Sigma A^2Bq^k\alpha^k/\{(1 - \alpha^2x)(1 - qy)\\
&\qquad\cdot(1 - qz)\} + \Sigma AB^2q^k\beta^k/\{(1 - \beta^2x)(1 - qy)(1 - qz)\}\\
&\qquad + B^3\beta^{3k}/\{(1 - \beta^2x)(1 - \beta^2y)(1 - \beta^2z)\}, \text{ by } (4)\\
&= f(x, y, z; k)/\ (1 - V_2x + q^2x^2)(1 - V_2y + q^2y^2)(1 - V_2z + q^2z^2)\\
&\qquad + ABq^k\Sigma(A\alpha^k(1 - \beta^2x) + B\beta^k(1 - \alpha^2x))/\{(1 - V_2x + q^2x^2)\\
&\qquad\cdot(1 - qy)(1 - qz)\}\\
&= d^2\cdot f(x, y, z; k)/D_W(x, y, z)\\
&\qquad + eq^k d^{-2}\Sigma(W_k - q^2W_{k-2}x)/\{(1 - V_2x + q^2x^2)(1 - qy)(1 - qz)\},
\end{aligned}$$

where
$$\begin{aligned}
f(x, y, z; k) &= A^3\alpha^{3k}(1 - \beta^2x)(1 - \beta^2y)(1 - \beta^2z)\\
&\qquad + B^3\beta^{3k}(1 - \alpha^2x)(1 - \alpha^2y)(1 - \alpha^2z)
\end{aligned}$$

$$= \sum_{j=0}^{3} (-q^2)^j h_j W(3,\ 3k - 2j).$$

From Lemma 1, we obtain

$$d^2 \cdot f(x,\ y,\ z;\ k) = \sum_{j=0}^{3} (-q^2)^j h_j \left\{ \sum_{r=0}^{2} \binom{2}{r} (-aq)^r b^{2-r} W_{3k-2j-r} \right\}$$
$$= W_3(x,\ y,\ z;\ k),$$

which proves Theorem 3.

Taking $k = 0$ in Theorem 3, we have

Corollary 4: $$\sum_{m,n,t=0}^{\infty} W_{m+n} W_{n+t} W_{t+m} x^m y^n z^t = W_3(x,\ y,\ z;\ 0)/D_W(x,\ y,\ z) + ed^{-2}\Sigma(a - q^2 W_{-2}x)/\{(1 - V_2 x + q^2 x^2)(1 - qy)(1 - qz)\}.$$

Obviously, all formulas of §4 in [1] are special cases of Theorem 3 and Corollary 4.

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