LIMITS OF q-POLYNOMIAL COEFFICIENTS

KUNG-WEI YANG
Western Michigan University, Kalamazoo, MI 49008

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INTRODUCTION

It is well known that the q-binomial (Gaussian) coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$ satisfy the "finite" Euler identity ([2], p. 101):

$$\prod_{n-1 \geqslant i \geqslant 0} (1 + q^i x) = 1 + \sum_{n \geqslant r \geqslant 1} \begin{bmatrix} n \\ r \end{bmatrix} q^{\binom{r}{2}} x^r,$$

and that their q-adic limits

$$\lim_{n \to \infty} \begin{bmatrix} n \\ r \end{bmatrix} = \prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1}$$

satisfy the "infinite" Euler identity ([1], p. 254; [2], p. 105):

$$\prod_{i \geqslant 0} (1 + q^i x) = 1 + \sum_{r \geqslant 1} \prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1} q^{\binom{r}{2}} x^r.$$

In [5], we showed that the q-polynomial coefficients $\begin{bmatrix} n.m \\ r \end{bmatrix}$ satisfy the generalized "finite" Euler identity:

$$\prod_{n-1 \geqslant i \geqslant 0} \left(\sum_{m \geqslant j \geqslant 0} q^{ijm + \binom{j}{2}} x^j \right) = 1 + \sum_{nm \geqslant r \geqslant 1} \begin{bmatrix} n.m \\ r \end{bmatrix} q^{\binom{r}{2}} x^r.$$

We now complete the analogy by showing that the q-adic limits of these q-polynomial coefficients $G_r^{(m)}$ (for each $m \geqslant 1$) satisfy a recurrence relation which generalizes that satisfied by

$$\prod_{r \geqslant i \geqslant 1} (1 - q^i)^{-1},$$

and the generalized infinite Euler identity:

$$\prod_{i \geqslant 0} \left(\sum_{m \geqslant j \geqslant 0} q^{ijm + \binom{j}{2}} x^j \right) = 1 + \sum_{r \geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

This paper is organized as follows. We begin in Section 1 by defining the basic graphical terms. We then make the first of two valuations of the digraph in Section 2. In Section 3, the recurrence formula for $G_r^{(m)}$ is proved. The generalized infinite Euler identity is proved in Section 4, and Section 5 contains a short discussion of the special cases $m = 1$ and $m = 2$.

We recall here the definition of the q-polynomial coefficients (see [4], [5], and [6]). Let $(m_1, \ldots, m_n)$ denote the *multiset* on $\{1, \ldots, n\}$ in which the multiplicity of i is m_n. The number of elements in $(m_1, \ldots, m_n)$ is $m_1 + \cdots + m_n$ and is denoted by $|(m_1, \ldots, m_n)|$. We abbreviate the multiset $(m_1, \ldots, m_n)$ in which $m_1 = \cdots = m_n = m$ to $(n.m)$. A *multisubset* $(a_1, \ldots, a_n)$ of $(n.m)$ satisfies $a_i \leqslant m$, for $i = 1, \ldots, n$, and it uniquely determines a *complementary multisubset* $(a_1', \ldots, a_n')$ satisfying $a_i + a_i' = m$ $(i = 1, \ldots, n)$. An *inversion* between the multisets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, in that order, is a pair (i, j), where i is an element of the multiset $(a_1, \ldots, a_n)$ and j is an element of $(b_1, \ldots, b_n)$, and $i > j$. Let $I(a_1, \ldots, a_n)$ denote the number of inversions between $(a_1, \ldots, a_n)$ and $(a_1', \ldots, a_n')$, where $(a_1, \ldots, a_n)$ is a multisubset of $(n.m)$. The *q-polynomial coefficient* $\begin{bmatrix} n.m \\ r \end{bmatrix}$ is defined to be the generating function

$$\begin{bmatrix} n.m \\ r \end{bmatrix} = \sum_{|(a_1, \ldots, a_n)| = r} q^{I(a_1, \ldots, a_n)}.$$

1. GRAPHS

Let m be a fixed positive integer. We consider the digraph with vertices all the lattice points in the first quadrant of the plane

$$\{(i, j) \mid i, j \geqslant 0\}$$

and directed edges

$$(i, j) \to (i + 1, j), \ (i, j) \to (i, j + 1) (i, j \geqslant 0).$$

We will call a vertex an *m-vertex* if there is a nonnegative integer k such that $i + j = km$. We will call a path of the form

$$\begin{aligned}(i, j) &\to (i + 1, j) \to \cdots \to (i + a, j) \\ &\to (i + a, j + 1) \to \cdots \to (i + a, j + b),\end{aligned}$$

where (i, j) is an m-vertex and $a + b = m$, an *m-arc*, and we will denote it by

$$(i, j) \to\to (i + a, j + b).$$

An m-arc of the form $(i, j) \to\to (i, j + m)$ will be called a *vertical m-arc*.

A finite sequence of consecutive m-arcs beginning with the origin followed by an infinite sequence of consecutive vertical m-arcs is called an *m-path*. In an m-path, if $(r - a, s - b) \to\to (r, s)$, where $a + b = m$, is the *last* nonvertical m-arc, (r, s) will be called the *terminal m-vertex* of the m-path. The part of an m-path between (0, 0) and its terminal m-vertex will be called the *valuable part* of the m-path.

2. VALUATION

Until Section 4, we will assign to all directed edges of the form $(i, j) \to (i + 1, j)$ the monomial $q^j x$ and directed edges of the form $(i, j) \to (i, j + 1)$ the trivial monomial 1 $(i, j \geqslant 0)$.

The product of all the monomials on the m-path p (m-arc) is then called the *value* of the m-path p (m-arc) and is denoted by $v(p; q, x)$. Clearly, the value of an m-path is completely determined by its valuable part. In fact, if (r, s) is the terminal m-vertex, and if

$$(0, 0) \to\to (a_1, a_1') \to\to (a_1 + a_2, a_1' + a_2') \to\to \cdots \to\to (a_1 + \cdots + a_n, a_1' + \cdots + a_n') = (r, s)$$

is the valuable part of the m-path, the value of the m-path p is

$$v(p; q, x) = q^{a_2 a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}')} x^r.$$

Observe that

$$I(a_1, \ldots, a_n) = a_2 a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}').$$

This shows $v(p; q, x) = q^{I(a_1, \ldots, a_n)} x^r$. Hence,

Lemma 1: $\begin{bmatrix} n.m \\ r \end{bmatrix} = \sum v(p; q, 1)$, where the sum is over all m-paths from (0, 0) to $(r, nm - r)$.

We note that $I(a_1, \ldots, a_n)$ is also equal to the number of unit squares (area) under the m-path p ([3], p. 13).

Theorem 1: Keeping the above notation, we have

$$I(a_1, \ldots, a_n) = I(a_n', \ldots, a_1').$$

Proof:
$$\begin{aligned} I(a_n', \ldots, a_1') &= a_{n-1}' a_n + a_{n-2}'(a_n + a_{n-1}) + \cdots + a_1'(a_n + \cdots + a_2) \\ &= a_2 a_1' + a_3(a_1' + a_2') + \cdots + a_n(a_1' + \cdots + a_{n-1}') \\ &= I(a_1, \ldots, a_n). \quad \text{Q.E.D.} \end{aligned}$$

3. RECURRENCE RELATIONS

Let $G^{(m)}(q, x)$ denote the power series obtained from summing the value of all the m-paths. Writing in the ascending powers of x,

$$G^{(m)}(q, x) = 1 + \sum_{r \geqslant 1} G_r^{(m)} x^r,$$

we see that $G_r^{(m)} = \sum v(p; q, 1)$, where the sum is over the set of m-paths with terminal m-vertex on the line $x = r$. Lemma 1 now implies

Corollary 2: $\begin{bmatrix} n.m \\ r \end{bmatrix} \to G_r^{(m)}$, as $n \to \infty$.

Theorem 3: Let $G_0^{(m)} = 1$, $G_r^{(m)} = 0$, if $r < 0$. Then, for all $r > 1$,

$$G_r^{(m)} = (1 - q^{rm})^{-1}\left(\sum_{m \geqslant i \geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right).$$

Proof: Let p be an m-path with terminal m-vertex on the line $x = r$. Choose the largest k such that $(0, km)$ is an m-vertex of p and let $(i, (k + 1)m - i)$ be the next m-vertex, $1 \leqslant i \leqslant m$. Then

$$v(p; q, 1) = q^{rkm + (r-i)(m-i)} v(p'; q, 1),$$

where p' is the m-path obtained by deleting the part from $(0, 0)$ to $(i, (k + 1)m - i)$ from p and then translating so that the starting point is at the origin. The sum of $v(p'; q, 1)$ for all such p' is $G_{r-i}^{(m)}$. Thus,

$$\begin{aligned} G_r^{(m)} &= \sum_{k \geqslant 0} q^{rkm}\left(\sum_{m \geqslant k \geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right) \\ &= (1 - q^{rm})^{-1}\left(\sum_{m \geqslant i \geqslant 1} q^{(r-i)(m-i)} G_{r-i}^{(m)}\right). \quad \text{Q.E.D.} \end{aligned}$$

4. IDENTITIES

Now, we multiply an additional factor of q^i to each monomial $q^j x$ already assigned to the directed edges between the lines $x = i$ and $x = i + 1$. Thus, the total sum of the values of all the m-paths is clearly changed from

$$1 + \sum_{r \geqslant 1} G_r^{(m)} x^r$$

to

$$1 + \sum_{r \geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

On the other hand, the sum of the values of the m-arcs emanating from *each* m-vertex (r, s) satisfying $r + s = im$ is now uniformly equal to

$$\sum_{m \geqslant j \geqslant 0} q^{ijm + \binom{j}{2}} x^j.$$

Since each m-path consists of a valuable part followed by an infinite sequence of consecutive vertical m-arcs the value of which is 1, and since the valuable part consists of a finite sequence of consecutive m-arcs starting with (0, 0) and ending at its terminal m-vertex, the total sum of the values of the m-paths is equal to

$$\prod_{i\geqslant 0}\left(\sum_{m\geqslant j\geqslant 0} q^{ijm+\binom{j}{2}}x^j\right).$$

Equating these two formal power series and invoking Corollary 2, we obtain

Theorem 4: Let $G_r^{(m)}$ be the q-adic limit of $\begin{bmatrix} n.m \\ r \end{bmatrix}$ as $n \to \infty$. Then they satisfy

$$\prod_{i\geqslant 0}\left(\sum_{m\geqslant j\geqslant 0} q^{ijm+\binom{j}{2}}x^j\right) = 1 + \sum_{r\geqslant 1} G_r^{(m)} q^{\binom{r}{2}} x^r.$$

It should be noted that Theorem 4 also follows directly from Theorem 3.

5. SPECIAL CASES

The case $m = 1$ is, of course, the Euler identity:

$$\prod_{i\geqslant 0}(1 + q^i x) = 1 + \sum_{r\geqslant 1} G_r^{(1)} q^{\binom{r}{2}} x^r,$$

where $G_0^{(1)} = 1$, and $G_r^{(1)} = \prod_{r\geqslant i\geqslant 1}(1 - q^i)^{-1}$, if $r \geqslant 1$.

When $m = 2$, the recurrence for $G_r^{(2)}$ is

$$G_r^{(2)} = (1 - q^{2r})^{-1}q^{r-1}G_{r-1}^{(2)} + (1 - q^{2r})^{-1}G_{r-2}^{(2)},$$

where $G_0^{(2)} = 1$, $G_{-1}^{(2)} = 0$. If we let r be $\geqslant 1$, $a_{r-1} = (1 - q^{2r})^{-1}q^{r-1}$, and $b_{r-2} = (1 - q^{2r})^{-1}$, the recurrence can be written as

$$G_r^{(2)} = a_{r-1}G_{r-1}^{(2)} + b_{r-2}G_{r-2}^{(2)}.$$

Using this notation, we may write the infinite product identity for the case $m = 2$ as

$$\begin{aligned}
&(1 + x + qx^2)(1 + q^2x + q^5x^2)\ \ldots\ (1 + q^2\ x + q^{4r+1}x^2)\ \ldots \\
&= 1 + a_0 q^{\binom{1}{2}}x + (a_0a_1 + b_0)q^{\binom{2}{2}}x^2 + (a_0a_1a_2 + b_0a_2 + a_0b_1)q^{\binom{3}{2}}x^3 \\
&\quad + (a_0a_1a_2a_3 + b_0a_2a_3 + a_0b_1a_3 + a_0a_1b_2 + b_0b_1)q^{\binom{4}{2}}x^4 \\
&\quad + \cdots + \left(\sum_{a_ia_{i+1}|\to b_i} a_0a_1\ \ldots\ a_{r-1}\right)q^{\binom{r}{2}}x^r + \cdots \\
&= 1 + (1 - q^2)^{-1}q^{\binom{1}{2}}x + \{(1 - q^2)^{-1}q(1 - q^4)^{-1} + (1 - q^4)^{-1}\}q^{\binom{2}{2}}x^2
\end{aligned}$$

(continued)

$$+ \{(1 - q^2)^{-1}q(1 - q^4)^{-1}q^2(1 - q^6)^{-1} + (1 - q^4)^{-1}q^2(1 - q^6)^{-1}$$
$$+ (1 - q^2)^{-1}(1 - q^6)^{-1}\}q^{\binom{3}{2}}x^3 + \cdots .$$

Here, by the notation,

$$\sum_{a_i a_{i+1}| \to b_i} a_0 a_1 \ldots a_{r-1}$$

we mean that the sum is over all possible products obtainable from $a_0 a_1 \ldots a_{r-1}$ by replacing in it blocks of two consecutive $a_i a_{i+1}$ by b_i. There are F_r (Fibonacci number) such formal terms in $G_r^{(2)}$. This can be seen, by induction, from

$$\begin{aligned} G_r^{(2)} &= a_{r-1}G_{r-1}^{(2)} + b_{r-2}G_{r-2}^{(2)} \\ &= \left(\sum_{a_i a_{i+1}| \to b_i} a_0 a_1 \ldots a_{r-2}\right)a_{r-1} + \left(\sum_{a_i a_{i+1}| \to b_i} a_0 a_1 \ldots a_{r-3}\right)b_{r-2} \\ &= \sum_{a_i a_{i+1}| \to b_i} a_0 a_1 \ldots a_{r-1}. \end{aligned}$$

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