ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-421 *Proposed by Piero Filipponi, Rome, Italy*

Let the numbers $U_n(m)$ (or merely U_n) be defined by the recurrence relation [1]

$$U_{n+2} = mU_{n+1} + U_n; \quad U_0 = 0,\ U_1 = 1,$$

where $m \in N = \{1, 2, \ldots\}$.

Find a compact form for

$$S(k, h, n) = \sum_{j=0}^{n-1} U_{k+jh} U_{k+(n-1-j)h} \quad (k, h, n \in N).$$

Note that, in the particular case $m = 1$, $S(1, 1, n) = F_n^{(1)}$ is the n^{th} term of the Fibonacci first convolution sequence [2].

References

1. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *The Fibonacci Quarterly 13*, no. 4 (1975):345-349.
2. V. E. Hoggatt, Jr. "Convolution Triangles for Generalized Fibonacci Numbers." *The Fibonacci Quarterly 8*, no. 2 (1970):158-171.

H-422 *Proposed by Larry Taylor, Rego Park, NY*

(A1) Generalize the numbers (2, 2, 2, 2, 2, 2, 2) to form a seven-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A2) Generalize the numbers (1, 1, 1, 1, 1, 1) to form a six-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference F_n.

(A3) Generalize the numbers (4, 4, 4, 4, 4) to form a five-term arithmetic progression of integral multiples of Fibonacci and/or Lucas numbers with common difference $5F_n$.

(A4) Generalize the numbers (3, 3, 3, 3), (3, 3, 3, 3), (3, 3, 3, 3) to form three four-term arithmetic progressions of integral multiples of Fibonacci and/or Lucas numbers with common differences F_n, $5F_n$, F_n, respectively.

(B) Generalize the Fibonacci and Lucas numbers in such a way that, if the Fibonacci numbers are replaced by the generalized Fibonacci numbers and the Lucas numbers are replaced by the generalized Lucas numbers, the arithmetic progressions still hold.

SOLUTIONS

Late Acknowledgment: *C. Georghiou solved H-394.*

A Simple Sequence

H-400 *Proposed by Arne Fransen, Stockholm, Sweden*
(Vol. 24, no. 3, August 1986)

For natural numbers h, k, with k odd, and an irrational a in the Lucasian sequence $V_{kh} = a^{kh} + a^{-kh}$, define $y_k \equiv V_{kh}$. Put

$$y_k = \sum_{r=0}^{n} c_r^{(2n+1)} y_1^{(2r+1)}, \text{ with } k = 2n+1.$$

Prove that the coefficients are given by

$$c_r^{(2n+1)} \begin{cases} \equiv 1 & \text{for } r = n, \\ = (-1)^{n-r}(2n+1)\displaystyle\sum_{j=1}^{J} \frac{1}{2j-1}\binom{n-j}{2(j-1)}\binom{n-1-3(j-1)}{r-(j-1)} & \text{for } 0 \leqslant r < n, \end{cases}$$

where $J = \min\left(\left[\dfrac{n+2}{3}\right], \left[\dfrac{n+1-r}{2}\right], r+1\right)$.

Also, is there a simpler expression for $c_r^{(2n+1)}$?

Solution by Paul Bruckman, Fair Oaks, CA

Let $a^h = e^{i\theta}$, so that

$$y_k = 2\cos k\theta. \tag{1}$$

Examining the Chebyshev polynomials of the first kind (viz. 22.3.15 of [1]), we find the following relation:

$$T_m(\cos\theta) = \cos m\theta, \quad m = 1, 2, 3, \ldots, \tag{2}$$

where (22.3.6, *ibid.*)

$$T_m(x) = \sum_{r=0}^{[\frac{1}{2}m]} \tfrac{1}{2}m(-1)^r \frac{\binom{m-r}{r}}{m-r}(2x)^{m-2r}. \tag{3}$$

Substitute $x = \cos\theta$, $m = k = 2n+1$ in (3). Then, from (2) and (1),

$$\cos k\theta = \tfrac{1}{2}y_k = \sum_{r=0}^{n} \tfrac{1}{2}k(-1)^r \frac{\binom{k-r}{r}}{k-r} y_1^{k-2r};$$

further substituting $n - r$ for r gives

$$y_k = k\sum_{r=0}^{n}(-1)^{n-r}\frac{\binom{n+r}{2r}}{2r+1}y_1^{2r+1}. \tag{4}$$

It follows that we have obtained the desired simple expression:

$$c_r^{(k)} = \frac{k}{2r+1}(-1)^{n-r}\binom{n+r}{2r}. \tag{5}$$

Note the following:

$$c_n^{(k)} = 1. \tag{6}$$

Let the given alleged expression for $c_r^{(k)}$ be denoted by $b_r^{(k)}$. Thus,

$$b_r^{(k)} \equiv (-1)^{n-r}k\sum_{j=0}^{J-1}\frac{1}{2j+1}\binom{n-1-j}{2j}\binom{n-1-3j}{r-j},\ 0 \leqslant r < n. \tag{7}$$

Note that the conditions $2j \leqslant n - 1 - r$, $j \leqslant r$ imply $3j \leqslant n - 1$; hence,

$$J - 1 = \min([\tfrac{1}{2}(n-1-r)], r).$$

After some manipulation, we obtain

$$b_r^{(k)} = (-1)^{n-r}\frac{k}{n-r}\sum_{j=0}^{J-1}\binom{n-r}{2j+1}\binom{n-1-j}{r-j}. \tag{8}$$

To sum (8), we use the following combinatorial identity (viz. 3.25 in [2]):

$$\sum_{j=0}^{r}\binom{x}{2j+1}\binom{x+r-j-1}{r-j} = \binom{x+2r}{2r+1}. \tag{9}$$

Let $x = n - r$ in (9). Note that terms for which $n - r < 2j + 1$ vanish, so $j \leqslant [\frac{1}{2}(n - 1 - r)]$; also, $j \leqslant r$. Thus, (9) becomes

$$\sum_{j=0}^{J-1}\binom{n-r}{2j+1}\binom{n-1-j}{r-j} = \binom{n+r}{2r+1}. \tag{10}$$

Comparison with (8) yields $b_r^{(k)} = \dfrac{k}{n-r}(-1)^{n-r}\dbinom{n+r}{2r+1}$, or

$$b_r^{(k)} = \frac{k}{2r+1}(-1)^{n-r}\binom{n+r}{2r},\ 0 \leqslant r < n. \tag{11}$$

Comparison of (5) and (11) yields the desired relation:

$$b_r^{(k)} \equiv c_r^{(k)},\ 0 \leqslant r < n. \quad \text{Q.E.D.} \tag{12}$$

References

1. M. Abramowitz & I.A. Stegun, eds. *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables.* 9th printing. National Bureau of Standards, 1970.
2. H.W. Gould. *Combinatorial Identities.* Morgantown, West Virginia, 1972.

Fibonacci in His Prime

H-401 *Proposed by Albert A. Mullin, Huntsville, AL*
(Vol. 24, no. 3, August 1986)

It is well known that, if $n \neq 4$ and the Fibonacci number F_n is prime, then n is prime.

(1) Prove or disprove the complementary result: If $n \neq 8$ and the Fibonacci number F_n is the product of two *distinct* primes then n is either prime or the product of two primes, in which case at least one prime factor of F_n is Fibonacci.

(2) Define the recursions $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Prove or disprove that each sequence $\{u_n\}$ represents only finitely many primes and finitely many products of two distinct primes.

Solution by Lawrence Somer, Washington, D.C.

(1) The result is true. It was proved in both [3] and [4] that F_n is the product of two distinct primes only if $n = 8$ or n is of the form p, $2p$, or p^2, where p is an odd prime. It is well known that if $m|n$, then $F_m|F_n$. A prime p is called a primitive divisor of F_n if $p|F_n$, but $p \nmid F_m$ for $0 < m < n$. In [1], R. Carmichael proved that F_n has a primitive prime divisor for every n except $n = 1, 2, 6$, or 12. If $n = 1, 2, 6$, or 12, then F_n is not the product of two distinct primes. It thus follows that if $n > 6$ and n is of the form $2p$ or p^2, then F_n has at least two distinct prime divisors—one of the primitive prime divisors of F_p and one of the primitive prime divisors of F_n. Clearly, every prime divisor of F_p is a primitive divisor. Thus, if F_n is the product of two distinct primes and $n = 2p$ or $n = p^2$, then F_p must be a prime divisor of F_n. The result now follows.

(2) As stated by the proposer, if $n \neq 4$, then F_n can be prime only if n is prime. Thus, it is conceivable that if $p > 6$, p is a prime, and $u_1 = F_p$ is prime, then u_n is prime for all n, and $\{u_n\}$ represents infinitely many primes. However, if u_n is not prime for some n, then we claim that, for any fixed positive integer k, there exist only finitely many positive integers n such that u_n has exactly k distinct prime divisors. In particular, $\{u_n\}$ represents only finitely many products of two distinct primes no matter what u_1 is. In fact, the following theorem and corollary are true.

Theorem: Let $\{u_n\}$ be defined by $u_{n+1} = F_{u_n}$, $u_1 = F_m$, $m \geqslant 6$. Let $d(u_n)$ denote the number of distinct prime divisors of u_n, then $d(u_{n+1}) \geqslant d(u_n)$. If $d(u_n) = r \geqslant 3$, then

$$d(u_{n+1}) \geqslant 2^r - 3 > d(u_n).$$

If $d(u_n) = 2$ and if it is not the case that both $n = 1$ and $u_n = F_9 = 34$, then $d(u_{n+1}) \geqslant 3 > d(u_n)$. If $u_n = F_9 = 34$, then $n = 1$ and $d(u_{n+1}) = 2 = d(u_n)$. If $d(u_n) = 1$ and $u_n = p^s$, where p is an odd prime and $s \geqslant 1$, then $d(u_{n+1}) \geqslant s$. If $d(u_n) = 1$ and $u_n = 2^s$, where $s \geqslant 2$, then $d(u_{n+1}) \geqslant s - 1$.

Corollary: Let t be the least positive integer, if it exists, such that u_t is not a prime. Then $\{u_n\}$ represents exactly $t - 1$ primes and at most t integers that are prime powers. If such a positive integer t does not exist, then $\{u_n\}$ represents infinitely many primes and only primes. For a fixed integer $k \geqslant 3$, $\{u_n\}$ represents at most one integer having exactly k distinct prime divisors. If $u_1 \neq 34 = F_9$, then $\{u_n\}$ represents at most one integer having exactly two prime divisors. If $u_1 = 34 = F_9$, then $\{u_n\}$ represents exactly two integers having exactly two distinct prime divisors.

Proof of the Theorem: By Carmichael's result in [2] stated earlier, F_n has a primitive prime divisor if $n \neq 1, 2, 6$, or 12. Suppose $d(u_n) = r \geqslant 3$. Then u_n has 2^r distinct divisors that are products of distinct primes or equal to 1. If k is a divisor of u_n which is the product of distinct primes and if $k \neq 1$, 2, or 6, then $F_k | F_{u_n}$ and F_k has at least one primitive prime divisor. It thus follows that $d(u_{n+1}) \geqslant 2^r - 3 > d(u_n) = r$.

Now suppose $d(u_n) = 2$ and $u_n \neq F_9 = 34$. We claim that $d(u_{n+1}) \geqslant 3$. First we prove that if $d(u_n) = 2$, $u_n \neq F_9 = 34$, and $u_n \neq F_{12} = 144$, then $2 \nmid u_n$. If $2 | F_j$, then it is known that $3 | j$. If $j = 3i$, where $i \geqslant 5$, then F_j is divisible by F_3, F_i, and F_{3i}, each of which has a primitive prime divisor. Thus, F_{3i}, $i \geqslant 5$, has at least three distinct prime divisors. The result now follows because F_3 and F_6 do not have exactly two distinct prime divisors. Thus, u_n has exactly two distinct odd prime divisors p and q. Then u_{n+1} is divisible by F_p, F_q, and F_{pq}, each of which has a primitive prime divisor. Hence, we have

$$d(u_{n+1}) \geqslant 3 > d(u_n) = 2.$$

If $u_n = F_{12} = 144$, then $u_{n+1} = F_{144}$. By the table given in [1, p. 8], $d(F_{144}) = 11$, and the claim follows. Now suppose $u_n = F_9 = 34$. Since 9 is not a Fibonacci number, we must have that $n = 1$. By the table given in [1, p. 2],

$$u_{n+1} = F_{34} = 5702887 = 1597.3571,$$

and $d(u_{n+1}) = 2 = d(u_n)$.

Now consider the case in which $d(u_n) = 1$ and $u_n = p^s$, where p is an odd prime and $s \geqslant 1$. Then u_{n+1} is divisible by F_{p^i} for $1 \leqslant i \leqslant s$, each of which has a primitive prime divisor. Hence, $d(u_{n+1}) \geqslant s$. Finally, suppose $d(u_n) = 1$ and $u_n = 2^s$, where $s \geqslant 2$. Then u_{n+1} is divisible by F_{2^i} for $2 \leqslant i \leqslant s$, each of which has a primitive prime divisor. Consequently,

$$d(u_{n+1}) \geqslant s - 1. \blacksquare$$

Proof of the Corollary: This follows immediately from the proof of the Theorem above upon noting that $u_{n+1} > u_n$ and that F_n is a power of 2 only in the cases $F_3 = 2$ and $F_6 = 8 = 2^3$. $\blacksquare$

References

1. Brother Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables*. Santa Clara, Calif.: The Fibonacci Association, 1972.
2. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $\alpha^n + \beta^n$." *Annals of Mathematics*, 2nd Ser. *15* (1913):30-70.
3. L. Somer. Solution to Problem B-456, proposed by A. A. Mullin. *The Fibonacci Quarterly 20*, no. 3 (1982):283.
4. L. Somer. Solution to Problem H-345, proposed by A. A. Mullin. *The Fibonacci Quarterly 22*, no. 1 (1984):92-93.

Also solved or partially solved by P. Bruckman, J. Desmond, and L. Kuipers.

Just a Game

H-402 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 24, no. 3, August 1986)

A MATRIX GAME (from the Italian TV serial *Pentathlon*).

For complete details of this very interesting problem, see pages 283-84 of *The Fibonacci Quarterly 24*, no. 3 (August 1986).

Solution by Paul S. Bruckman, Fair Oaks, CA

Given $n \geqslant 1$, let χ_n denote the set of $1 \times n$ vectors $(\theta_1, \theta_2, \ldots, \theta_n)$, χ_n' the set of $n \times 1$ vectors $(\theta_1, \theta_2, \ldots, \theta_n)'$ with $\theta_i = 0$ or 1 (chosen randomly). Let $T_n = \chi_n^n = \chi_n'^n$ denote the set of all $n \times n$ matrices with entries either 0 or 1. Let $\underline{\delta}_n \equiv (0, 0, \ldots, 0) \in \chi_n$, $\underline{\varepsilon}_n \equiv (1, 1, \ldots, 1) \in \chi_n$; likewise, $\underline{\delta}_n' \equiv (0, 0, \ldots, 0)' \in \chi_n'$, $\underline{\varepsilon}_n' \equiv (1, 1, \ldots, 1)' \in \chi_n'$. Let $\rho_n \equiv \{\underline{\delta}_n, \underline{\varepsilon}_n\}$, $\rho_n' \equiv \{\underline{\delta}_n', \underline{\varepsilon}_n'\}$; $\sigma_n \equiv \{\underline{\delta}_n, \underline{\delta}_n'\}$, $\tau_n \equiv \{\underline{\varepsilon}_n, \underline{\varepsilon}_n'\}$. We say a matrix *contains* a vector if the vector is either a row or a column, as appropriate, of the matrix.

Let A_n denote the subset of T_n containing at least one element of $\rho_n \cup \rho_n'$;
Let B_n denote the subset of T_n containing at least one element of ρ_n;
Let C_n denote the subset of T_n containing at least one element of ρ_n';
Let D_n denote the subset of T_n containing at least one element of ρ_n, ρ_n'.

We first observe that $|T_n| = 2^{n^2}$. Moreover,

$$P_n = |A_n|/|T_n| = 2^{-n^2}|A_n|. \tag{1}$$

By symmetry, we see that $|B_n| = |C_n|$. Also, $|A_n| = |B_n| + |C_n| - |D_n|$, so

$$|A_n| = 2|B_n| - |D_n|. \tag{2}$$

To evaluate $|B_n|$, we note that B_n^* is the subset of T_n containing no elements of ρ_n. Since each such (row) element of B_n^* may be chosen in $2^n - 2$ ways, thus, $|B_n^*| = (2^n - 2)^n$. Hence,

$$|B_n| = 2^{n^2} - (2^n - 2)^n. \tag{3}$$

To evaluate $|D_n|$, we first partition D_n into the two (disjoint) sets $D_n^{(0)}$ and $D_n^{(1)}$, defined as follows: $D_n^{(0)}$ is the subset of T_n containing σ_n, $D_n^{(1)}$ is the subset of T_n containing τ_n. Note that no element of T_n can contain $\{\underline{\delta}_n, \underline{\varepsilon}_n'\}$ or $\{\underline{\delta}_n', \underline{\varepsilon}_n\}$. By symmetry, $|D_n^{(0)}| = |D_n^{(1)}|$. Therefore,

$$|D_n| = 2|D_n^{(0)}|. \tag{4}$$

To evaluate $|D_n^{(0)}|$, we further partition $D_n^{(0)}$ into the (disjoint) sets $D_{n,k}^{(0)}$, $k = 1, 2, \ldots, n$, where $D_{n,k}^{(0)}$ is the subset of $D_n^{(0)}$ with at least one $\underline{\delta}_n$, with $\underline{\delta}_n'$ in the k^{th} column, but with no $\underline{\delta}_n'$ in any of the preceding columns. Thus,

$$|D_n^{(0)}| = \sum_{k=1}^{n} |D_{n,k}^{(0)}|. \tag{5}$$

Now, $D_{n,1}^{(0)}$ is the subset of T_n with at least one $\underline{\delta}_n$ and with first column $\underline{\delta}_n'$; this is equivalent to the set difference $E - F$, where E is the subset of T_n with first column $\underline{\delta}_n'$, F is the subset of E containing no $\underline{\delta}_n$. We enumerate E by considering the rows of any matrix in E. Each such row must have 0 as its first element, with the other elements random. This involves 2^{n-1} choices for each such row; hence, $|E| = 2^{(n-1)n}$. $|F|$ is enumerated similarly, except that each row of any matrix in F must also not be $\underline{\delta}_n$. This involves $2^{n-1} - 1$ choices for each row of any matrix in F; hence, $|F| = (2^{n-1} - 1)^n$. Therefore,

$$|D_{n,1}^{(0)}| = 2^{(n-1)n} - (2^{n-1} - 1)^n. \tag{6}$$

Next, we evaluate $|D_{n,2}^{(0)}|$. $D_{n,2}^{(0)}$ is the subset of T_n with at least one $\underline{\delta}_n$, with the first column *not* $\underline{\delta}_n'$ and with second column $\underline{\delta}_n'$. Thus, $D_{n,2}^{(0)}$ is equivalent to the set difference $G - H$, where G is the subset of T_n with at least one $\underline{\delta}_n$ and second column $\underline{\delta}_n'$, H is the subset of G where both first and second columns are $\underline{\delta}_n'$. By symmetry, we see that $|G| = |D_{n,1}^{(0)}|$. To evaluate $|H|$, we see that H is the set difference $J - K$, where J is the subset of T_n with both first and second columns $\underline{\delta}_n'$, and K is the subset of J containing no $\underline{\delta}_n$. By similar reasoning, $|J| = 2^{(n-2)n}$, $|K| = (2^{n-2} - 1)^n$. Hence, $|H| = 2^{(n-2)n} - (2^{n-2} - 1)^n$, and so

$$|D_{n,2}^{(0)}| = 2^{(n-1)n} - (2^{n-1}-1)^n - \{2^{(n-2)n} - (2^{n-2}-1)^n\}. \tag{7}$$

A moment's reflection shows us where this general process leads us; first, however, we make the following convenient definition:

$$a_k = 2^{(n-k)n} - (2^{n-k}-1)^n,\ k = 0, 1, 2, \ldots, n. \tag{8}$$

We then find: $|D_{n,1}^{(0)}| = a_1$, $|D_{n,2}^{(0)}| = a_1 - a_2 = -\Delta a_1$, $|D_{n,3}^{(0)}| = a_1 - 2a_2 + a_3 = \Delta^2 a_1$, etc.; in general, we find

$$|D_{n,k}^{(0)}| = (-1)^{k-1}\Delta^{k-1}a_1,\ k = 1, 2, \ldots, n. \tag{9}$$

Therefore, by (5), $|D_n^{(0)}| = \sum_{k=1}^n (-1)^{k-1}\Delta^{k-1}a_1$. This expression can be slightly simplified as follows:

$$\sum_{k=1}^n (-1)^{k-1}\Delta^{k-1}a_1 = \sum_{k=1}^n (-1)^{k-1}\Delta^{k-1}(1+\Delta)a_0$$

$$= \sum_{k=1}^n (-1)^{k-1}\Delta^{k-1}a_0 - \sum_{k=1}^n (-1)^k\Delta^k a_0 = -(-1)^{k-1}\Delta^{k-1}a_0\Big|_1^{n+1} = a_0 - (-1)^n\Delta^n a_0.$$

In terms of the binomial expansion,

$$|D_n^{(0)}| = \sum_{k=1}^n \binom{n}{k}(-1)^{k-1}a_k. \tag{10}$$

We may also express $|B_n|$ in (3) in terms of a_1, since we see from (3) that $|B_n| = 2^n(2^{(n-1)n} - (2^{n-1}-1)^n)$, i.e.,

$$|B_n| = 2^n a_1. \tag{11}$$

Using (2), (4), (10), and (11), we therefore obtain:

$$|A_n| = 2\left(2^n a_1 - \sum_{k=1}^n \binom{n}{k}(-1)^{k-1}a_k\right). \tag{12}$$

Finally, from (1), we obtain the desired exact expression:

$$P_n = 2^{1-n^2}\left(2^n a_1 - \sum_{k=1}^n \binom{n}{k}(-1)^{k-1}a_k\right), \tag{13}$$

where the a_k's are given by (8).

After some computations, we obtain the following values from (13): $P_1 = 1$, $P_2 = .875$, $P_3 = 205/256 \doteq .8008$, as discovered by the proposer. However, we further obtain: $P_4 = 21{,}331/32{,}768 \doteq .6510$, $P_5 = 7{,}961{,}061/16{,}777{,}216 \doteq .4745$, $P_6 = 10{,}879{,}771{,}387/34{,}559{,}738{,}368 \doteq .3166$, $P_7 \doteq .1978$, $P_8 \doteq .1215$, $P_9 \doteq .0680$, and $P_{10} \doteq .0383$, all of which values are different from those published in the statement of the problem.

Nevertheless, the proposer's conjecture is correct, and is easily proved. Note, from (13), that $P_n < 2^{1-n^2}2^n a_1$. Also,

$$a_1 = 2^{n^2-n} - (2^{n-1}-1)^n = 2^{n^2-n}\{1 - (1 - 2^{1-n})^n\}$$

$$= 2^{n^2-n}\{1 - 1 + n\cdot 2^{1-n} - \cdots\} < n \cdot 2^{n^2-2n+1}.$$

Hence, $P_n < 2^{1+n-n^2}\cdot n \cdot 2^{n^2-2n+1}$, or

$$P_n < \frac{4n}{2^n}. \tag{14}$$

Clearly, $\lim_{n\to\infty} 4n \cdot 2^{-n} = 0$. Hence, $\lim_{n\to\infty} P_n = 0$. Q.E.D.

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