ON THE NUMBER OF SOLUTIONS OF THE DIOPHANTINE EQUATION $\binom{x}{p} = \binom{y}{2}$

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It is well known that the binomial coefficients are equal in the trivial cases

$$1 = \binom{n}{0} = \binom{m}{0} = \binom{k}{k},\ \binom{n}{k} = \binom{n}{n-k},\ \text{and } N = \binom{n}{k} = \binom{N}{1}$$

for any positive integers n, m, and k ($\leqslant n$). Apart from these cases, it is more difficult to decide whether there are infinitely many pairs of equal binomial coefficients or not.

The problem of equal binomial coefficients was studies by several authors (e.g., Singmaster [6], [7]; Lind [4]; Abbot, Erdös, & Hanson [1]). Recently, in an article in this *Quarterly*, Tovey [8] showed that the equation

$$\binom{n}{k} = \binom{n-1}{k+1} \tag{1}$$

has infinitely many solutions; furthermore, (1) holds if and only if

$$n = F_{2i}(F_{2i} + F_{2i-1}) \text{ and } k = F_{2i}F_{2i-1} - 1 \quad (i = 1, 2, \ldots),$$

where F_j denotes the j^{th} Fibonacci number. Another type of result was conjectured by W. Sierpinski and solved by Avanesov [2]: There are only finitely many pairs $(x; y)$ of natural numbers such that $\binom{x}{3} = \binom{y}{2}$. Avanesov proved that this holds only in the cases $(x; y) = (3; 2), (5; 5), (10; 16), (22; 56)$, and $(36; 120)$.

The purpose of this paper is to prove an extension of Sierpinski's conjecture. We shall show that the conjecture is true even if we exchange 3 for any odd prime.

Theorem: Let p ($\geqslant 3$) be a fixed prime. Then the Diophantine equation

$$\binom{x}{p} = \binom{y}{2} \tag{2}$$

has only finitely many positive integer x, y solutions.

We need the following lemmas for the proof of our theorem.

Lemma 1: Let $m \geqslant 2$ and $n \geqslant 3$ be rational integers and let $a_n \neq 0, a_{n-1}, \ldots, a_0$ and b be rational numbers. If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has at least 3 simple roots, then all integer solutions x, z of the Diophantine equation

$$f(x) = b \cdot z^m$$

satisfy $\max(|x|, |z|) < C$, where C is a number which is effectively computable in terms of $a_0, \ldots, a_{n-1}, a_n$, and b.

Proof: The lemma is known if the coefficients of $f(x)$ are integers and $b = 1$ (see, e.g., Baker [3]). If b and the coefficients are rational numbers, then there is an integer d ($\neq 0$) such that $d \cdot f(x)$ is a polynomial with integer coefficients and $d \cdot b$ is an integer. Thus, our equation can be written in the form

$$(bd)^{m-1}d \cdot f(x) = (bdz)^m$$

which, by the result mentioned above, has only finitely many integer solutions.

Lemma 2: Let $p \geqslant 3$ be a fixed prime number. Then all the roots of the polynomial

$$f(x) = x(x-1)(x-2) \cdots (x-(p-1)) + \frac{p!}{8}$$

are simple.

Proof: First, we assume that $p > 3$. We only have to prove that $f(x)$ and its derivative $f'(x)$ are relatively prime, since that implies the lemma.

Let us consider the polynomial

$$f_1(x) = x(x-1)(x-2) \cdots (x-(p-1)). \tag{3}$$

It is a polynomial of degree p with leading coefficient 1; furthermore, the number of the solutions of the congruence

$$f_1(x) \equiv 0 \pmod{p}$$

is p ($x \equiv 0, 1, \ldots, p-1$). So, as is well known,

$$f_1(x) \equiv x^p - x \pmod{p},$$

that is, $f_1(x)$ has the form

$$f_1(x) = x^p - x + p \cdot g_1(x), \tag{4}$$

where $g_1(x)$ is a polynomial of degree less than p and has integer coefficients (see, e.g., Theorem 2.22 in [5]).

Since $p \geqslant 5$, $p!/8$ is an integer and $p \mid (p!/8)$; so, by (3) and (4), the polynomial $f(x)$ and its derivative $f'(x)$ are of the form

$$f(x) = x^p - x + p \cdot g(x)$$

and

$$f'(x) = -1 + p \cdot h(x),$$

respectively, for some polynomials $g(x)$ and $h(x)$ with integer coefficients. It follows that

$$f(x) - x \cdot f'(x) = x^p + p \cdot (g(x) - x \cdot h(x))$$
$$= b_p x^p + b_{p-1}x^{p-1} + \cdots + b_0,$$

where the b_i's are integers. It can be easily checked that $b_p = 1 - p$ and that $b_0 = p!/8$. Furthermore, $p \nmid b_p$, $p | b_i$ for $i = 0, 1, \ldots, p - 1$ and $p^2 \nmid b_0$. So, by Eisenstein's irreducibility criterion, $f(x) - x \cdot f'(x)$ is an irreducible polynomial over the rational number field. Hence, $f(x)$ and $f'(x)$ are relatively prime. This proves the lemma in the case in which $p > 3$.

When $p = 3$, one can directly show that the roots of $f(x)$ are simply, which completes the proof of Lemma 2.

Proof of the Theorem: Let x and y be integers for which (2) holds. Then

$$\frac{y(y - 1)}{2} = \binom{x}{p};$$

thus, the equation

$$y^2 - y - 2\binom{x}{p} = 0$$

has a positive integer solution y. From this it follows that there is an integer z such that

$$8\binom{x}{p} + 1 = z^2.$$

Consequently, x and z satisfy the Diophantine equation

$$f(x) = x(x - 1)(x - 2) \cdots (x - (p - 1)) + \frac{p!}{8} = \frac{p!}{8} \cdot z^2. \tag{5}$$

However, by Lemma 2, the roots of the polynomial $f(x)$ are simple; therefore, by Lemma 1, (5) has only finitely many integer solutions x, z, and the Theorem is proved.

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Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

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FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortezza. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Forty-two abstracts on all branches of mathematics and science related to the Fibonacci numbers and their generalizations have been received. All contributed papers will appear subject to approval by a referee in the Conference Proceedings, which are expected to be published in 1989.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks have been limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Computer Science, South Dakota State University, P.O. Box 2201, Brookings, South Dakota 57007-0194.