ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-423 *Proposed by Stanley Rabinowitz, Littleton, MA*

Prove that each root of the equation

$$F_n x^n + F_{n+1}x^{n-1} + F_{n+2}x^{n-2} + \cdots + F_{2n-1}x + F_{2n} = 0$$

has absolute value near ϕ, the golden ratio.

H-424 *Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy*

Let F_n and P_n denote the Fibonacci and Pell numbers, respectively.

Prove that, if F_p is a prime ($p > 3$), then either $F_p | P_H$ or $F_p | P_{H+1}$, where $H = (F_p - 1)/2$.

SOLUTIONS

Editorial Notes: Andrzej Makowski has pointed out that H-287 was published in the *American Mathematical Monthly* as Problem S 3 [1979, 55] and the solution appeared in [1980, 136].

Chris Long solved H-211 by using a Lemma of Wolstenholme [*Quart. Jour. Math.* 5 (1862), 35-39].

Brush the Dust Off

H-152 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA (deceased) (Vol. 7, no. 1, February 1969)*

Let m denote a positive integer and F_n the n^{th} Fibonacci number. Further, let $\{c_k\}$, $k = 1$ to ∞, be the sequence defined by

$$\{c_k\} \equiv \{\underbrace{(F_n)^m, (F_n)^m, \ldots, (F_n)^m}_{2^{m-1}\text{ copies}}\};\ m,\ k = 1 \text{ to } \infty,$$

Prove that $\{c_k\}$ is complete; i.e., show that every positive integer n has at least one representation of the form

$$n = \sum_{k=1}^{p} \alpha_k c_k,$$

where p is a positive integer and

$$\alpha_i = 0 \text{ or } 1 \text{ if } k = 1, 2, \ldots, p - 1,$$
$$\alpha_p = 1.$$

Solution by Chris Long, student, Rutgers University, New Brunswick, NJ

First some preliminaries.

Lemma 1: Let $\{x_i\}$, $i = 1$ to ∞, be a nondecreasing sequence of positive integers with $x_1 = 1$. Then $\{x_i\}$ is complete if and only if

$$x_{p+1} \leqq 1 + \sum_{1}^{p} x_i, \text{ for } p = 1, 2, \ldots .$$

Proof: This is proven in J. L. Brown, Jr., "Note on Complete Sequences of Integers," *Amer. Math. Monthly* 67 (1960):557-560.

Lemma 2: $(f_n)^m f_{n-1} + f_n (f_{n-1})^m \leqq (f_{n-1})^{m+1} + (f_n)^{m+1}$ for all m, $n \geqq 1$.

Proof: Since for m, $n \geqq 1$,

$$(f_{n-1})^m (f_n - f_{n-1}) \leqq (f_n)^m (f_n - f_{n-1}) \Rightarrow (f_n)^m f_{n-1} + f_n (f_{n-1})^m \leqq (f_{n-1})^{m+1} + (f_n)^{m+1}.$$

Lemma 3: $(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m)$ for all m, $n \geqq 1$.

Proof: We have $f_{n+1} \leqq f_{n-1} + f_n$ for all $n \geqq 1$. If

$$(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m),$$

then, since $f_{n+1} = f_{n-1} + f_n$ and $f_{n+1} > 0$ for all $n \geqq 1$,

$$(f_{n+1})^m f_{n+1} \leqq 2^{m-1}((f_{n-1})^{m+1} + (f_n)^{m+1} + (f_n)^m f_{n-1} + f_n (f_{n-1})^m)$$
$$\leqq 2^m ((f_{n-1})^{m+1} + (f_n)^{m+1}) \quad \text{by Lemma 2.}$$

Hence, by induction, $(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m)$ for all m, $n \geqq 1$.

Since $c_1 = 1$, $\{c_k\}$ is complete if $c_{k+1} \leqq 1 + c_1 + \cdots + c_k$ for all $k \geqq 1$. Now, if $2 \leqq \alpha \leqq 2^{m-1}$, then we have, for $k = n2^{m-1} + \alpha$, that

$$c_k = (f_{n+1})^m \leqq 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m) + (\alpha - 1)(f_{n+1})^m$$
$$= 1 + c_1 + \cdots + c_k;$$

therefore, we need only prove the case for $\alpha = 1$, and this is equivalent to

$$(f_{n+1})^m \leqq 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m).$$

But by Lemma 3,

$$(f_{n+1})^m \leqq 2^{m-1}((f_{n-1})^m + (f_n)^m) \leqq 1 + 2^{m-1}((f_0) + \cdots + (f_n)^m)$$
$$= 1 + 2^{m-1}((f_1)^m + \cdots + (f_n)^m). \quad \text{Q.E.D.}$$

At Last

H-215 *Proposed by Ralph Fecke, North Texas State University, Denton, TX (Vol. 11, no. 2, April 1973)*

a. Prove

$$\sum_{i=n}^{n+2} 2^i P_i \equiv 0 \pmod 5$$

for all positive integers n; P_i is the i^{th} term of the Pell sequence,

$$P_i = 1,\ P_2 = 2,\ P_{n+1} = 2P_n + P_{n-1} \quad (n \geq 2).$$

b. Prove $2^n L_n \equiv 2 \pmod{10}$ for all positive integers n; L_n is the n^{th} term of the Lucas sequence.

Solution by Chris Long, student, Rutgers University, New Brunswick, NJ

a. Note that $2P_1 + 4P_2 + 8P_3 \equiv 4P_2 + 8P_3 + 16P_4 \equiv 0 \pmod 5$ and that

$$\begin{aligned} 2^{i+2}P_{i+2} + 2^{i+1}P_{i+1} + 2^iP_i &= 4(2^{i+1}P_{i+1} + 2^iP_i + 2^{i-1}P_{i-1}) \\ &\quad + 4(2^iP_i + 2^{i-1}P_{i-1} + 2^{i-2}P_{i-2}); \end{aligned}$$

hence, by induction,

$$2^{i+2}P_{i+2} + 2^{i+1}P_{i+1} + 2^iP_i \equiv 0 \pmod 5$$

for all positive integers n.

b. We have that $2L_1 \equiv 4L_2 \equiv 2 \pmod{10}$ and that

$$2^{n+2}L_{n+2} = 2(2^{n+1}L_{n+1} + 2(2^nL_n));$$

hence, by induction,

$$2^nL_n \equiv 2 \pmod{10}$$

for all positive integers n.

Middle Aged

H-306 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (deceased) (Vol. 17, no. 3, October 1979)*

(a) Prove that the system S,

$$a + b = F_p,\ b + c = F_q,\ c + a = F_r,$$

cannot be solved in positive integers if F_p, F_q, F_r, are positive Fibonacci numbers.

(b) Likewise, show that the system T,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + e = F_s,\ e + a = F_t,$$

has no solution under the same conditions.

(c) Show that if F_p is replaced by any positive non-Fibonacci integer, then S and T have solutions.

If possible, find necessary and sufficient conditions for the system U,

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + a = F_s,$$

to be solvable in positive integers.

Solution by Chris Long, student, Rutgers University, New Brunswick, NJ

It is unclear whether the F's are meant to be distinct or not; I will consider both possibilities in the following.

(a) Assume WLOG that F_q is the maximum of F_p, F_q, F_r. We have that $2a = F_p - F_q + F_r$. If the F's are distinct, we then have that $F_q \geqq F_p + F_r$; hence, $2a \leqq 0$. Therefore, S cannot be solved in positive integers if the F's are distinct. If the F's are not distinct, then this is false; e.g., take $a = b = c = 1$.

(b) This is similar to (a). Assume that F_q is the maximum of the F's. We have that

$$2a = F_p - F_q + F_r - F_s + F_t \text{ and } 2d = F_s - F_t + F_p - F_q + F_r;$$

if the F's are distinct, then $F_q \geqq F_p + F_r$, which gives us that

$$2a \leqq F_t - F_s \text{ and } 2d \leqq F_s - F_t.$$

Adding gives the contradiction that $2(a + d) \leqq 0$; therefore, T cannot be solved in positive integers if the F's are distinct. Again, if the F's are not distinct, this is false; e.g., take $a = b = c = d = e = 1$.

(c) This is false for both (a) and (b). Indeed, for system S replace F_p with 4 and let $F_q = 1$ and $F_r = 2$; these values imply that $2a = 5$. Similarly, for system T replace F_p with 4 and let $F_q = 1$, $F_r = 2$, $F_s = 3$, and $F_t = 5$; these values imply that $2a = 7$.

For system U, I claim that it is solvable in positive integers if and only if $F_p + F_r = F_q + F_s$ and F_p, F_q, F_s, $F_t \geqq 2$. Indeed, the necessity of the statement is obvious. For sufficiency, note that all possible solutions must be of the form

$$(a, b, c, d) = (t, F_p + t, F_r - F_s + t, F_s - t);$$

hence, all solutions with a, b, c, d positive integers are given by

$$\{(t, F_p + t, F_r - F_s + t, F_s - t) \mid \max(1, F_s - F_r + 1) \leqq t \leqq F_s - 1\}.$$

In particular, $t = F_s - 1$ yields a solution under the given conditions. It is also interesting to note that the F's cannot all be distinct, as this would imply that one of the F's was $\leqq 0$.

Close Ranks

H-403 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
(Vol. 24, no. 4, November 1986)

Given p, q real with $p \neq -1 - 2qk$, $k = 0, 1, 2, \ldots$, find a closed form expression for the continued fraction

$$\theta(p, q) \equiv p + \cfrac{p + q}{p + 2q + \cfrac{p + 3q}{p + 4q + \cdots}}. \tag{1}$$

HINT: Consider the *Confluent Hypergeometric* (or *Kummer*) function defined as follows:

$$M(a,\ b,\ z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!},\ b \neq 0,\ -1,\ -2,\ \ldots\ . \tag{2}$$

NOTE: $\theta(1,\ 1) = 1 + \cfrac{2}{3 + \cfrac{4}{5 + \cdots}}$, which was Problem H-394.

Solution by C. Georghiou, University of Patras, Greece

Take the confluent hypergeometric differential equation

$$zw'' + (b - z)w' - aw = 0 \tag{$*$}$$

Then, for $a \neq 0, -1, -2, \ldots$ and $b \neq 0, -1, -2, \ldots$, we have that

$$\frac{w}{w'} = \frac{b - z}{a} + \frac{z/a}{w'/w''}.$$

By differentiating ($*$), we get

$$\frac{w'}{w''} = \frac{b + 1 - z}{a + 1} + \frac{z/(a + 1)}{w''/w'''}$$

and by repeated differentiation of ($*$), we get the continued fraction

$$\frac{bw'}{aw} \equiv f(z) = \cfrac{b}{b - z + \cfrac{z}{\cfrac{b+1-z}{a+1} + \cfrac{z/(a+1)}{\cfrac{b+2-z}{a+2} + \cdots}}}$$

From the theory of continued fractions, we know that

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + \frac{c_1a_1}{c_1b_1} + \frac{c_1c_2a_2}{c_2b_2} + \frac{c_2c_3a_3}{c_3b_3} + \cdots \tag{$**$}$$

where $c_n \neq 0$, and setting $c_1 = 1$, $c_2 = a + 1$, ..., $c_n = a + n - 1$, ..., we get

$$f(z) = \frac{b}{b - z +}\ \frac{(a+1)z}{b + 1 - z +}\ \frac{(a+2)z}{b + 2 - z +}\ \cdots\ . \tag{$***$}$$

Now it is shown in W. B. Jones & W. J. Thron, "Continued Fractions," in G.-C. Rota, ed., *Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, 1980, pp. 276-282, that the above continued fraction converges to the meromorphic function

$$f(z) = \frac{M(a + 1,\ b + 1,\ z)}{M(a,\ b,\ z)}$$

for all complex numbers z and, moreover, the convergence is uniform on every compact subset of $\mathbb{C}$ which contains no poles of $f(z)$.

Before we proceed further, we note that the restriction $a \neq 0, -1, -2, \ldots$ can be removed by a limiting argument (see also the above-mentioned reference).

Now, for $b \neq 0, -1, -2, \ldots$, and $q \neq 0$ and $c_n = 2q$, $n = 1, 2, 3, \ldots$, ($**$) and ($***$) give

$$\frac{M(a+1,\, b+1,\, z)}{M(a,\, b,\, z)} = \frac{2qb}{2q(b-z)+}\;\frac{4q^2(a+1)z}{2q(b+1-z)+}\;\frac{4q^2(a+2)z}{2q(b+2-z)+}\;\cdots .$$

Finally, take $a = (p-q)/2q$, $b = (p+1)/2q$, and $z = 1/2q$. Then,

$$\frac{M\left(\dfrac{p+q}{2q},\, \dfrac{p+1+2q}{2q},\, \dfrac{1}{2q}\right)}{M\left(\dfrac{p-q}{2q},\, \dfrac{p+1}{2q},\, \dfrac{1}{2q}\right)} = \cfrac{p+1}{p+\cfrac{p+q}{p+2q+\cfrac{p+3q}{p+4q+\cdots}}} = \frac{p+1}{\theta(p,\, q)},$$

and the final result is

$$\theta(p,\, q) = (p+1)\,\frac{M\left(\dfrac{p-q}{2q},\, \dfrac{p+1}{2q},\, \dfrac{1}{2q}\right)}{M\left(\dfrac{p+q}{2q},\, \dfrac{p+1+2q}{2q},\, \dfrac{1}{2q}\right)}$$

valid for p, q such that $q \neq 0$ and $p \neq -1 - 2qk$, $k = 0, 1, 2, \ldots$.

Again the restriction $p \neq -1 - 2qk$ can be removed since it is easy to see that

$$\theta(-1-2qk,\, q)$$
$$= -1 - 2qk + \frac{-1-(2k-1)q}{-1-(2k-2)q+}\;\frac{-1-(2k-3)q}{-1-(2k-4)q+}\cdots\frac{-1-q}{-1}\;\frac{-1+q}{+\;\theta(-1+2q,\, q)}.$$

For example, for $k = 0$ (and $q = 0$), we have

$$\theta(-1,\, q) = -1 + \frac{-1+q}{\theta(-1+2q,\, q)} = -1 + \frac{q-1}{2q}\,\frac{M\left(\dfrac{3q-1}{2q},\, 2,\, \dfrac{1}{2q}\right)}{M\left(\dfrac{q-1}{2q},\, 1,\, \dfrac{1}{2q}\right)}$$

and the same result is obtained from the given expression of $\theta(p,\, q)$ by a limiting argument when $p \to -1$. The same is true for $k > 0$.

Finally, when $q = 0$, we have a periodic continued fraction and

$$\theta(p,\, 0) = p + \cfrac{p}{p+\cfrac{p}{p+\cdots}} = p + \frac{p}{\theta(p,\, 0)}$$

which gives for $p > 0$ or $p \leqq -4$

$$\theta(p,\, 0) = (p + \sqrt{p^2+4p})/2.$$

For $-4 < p \leqq 0$, $\theta(p,\, 0)$ diverges.

Also solved by the proposer.

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