A GENERALIZATION OF METROD'S IDENTITY

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In 1913, G. Métrod published an arithmetical identity involving Euler's function ϕ and the Jordan function J_2, and asked if a similar identity holds for other Jordan functions [2, p. 155]. The Jordan functions J_k, for $k = 1, 2, \ldots$, are the Dirichlet convolutions of the functions ζ_k and the Möbius function μ, where $\zeta_k(n) = n^k$ for all n, i.e., $J_k = \zeta_k * \mu$. Cohen [1] answered Métrod's question by showing that, for all n,

$$\sum_{d|n} \sum_{e|d} J_k(n/d) J_s(n/e) e^k = n^{k+s}. \tag{1}$$

Métrod's identity is the special case $k = 1$, $s = 2$.

H. Stevens [5] defined a class of arithmetical functions which includes the Jordan functions and he showed that a suitable identity analogous to (1) holds for any two functions in the class. All of Stevens' functions can be written in the form $g * h^{-1}$, where g and h are completely multiplicative, i.e., $g(mn) = g(m)g(n)$ for all m and n, and similarly for h. The function h^{-1} is the inverse of h with respect to Dirichlet convolution.

In this note, we point out that there is an identity which extends (1) and Stevens' identity in several ways. It involves an arbitrary finite number of functions, and the functions are not restricted as severely as those described above. Furthermore, it holds for an arbitrary regular arithmetical convolution. We shall derive the identity for the Dirichlet convolutions, and restate it in the more general setting at the end of the note. Our terminology and notation will be that used in [3].

For $i = 1, \ldots, k$, let $f_i = g_i * h_i^{-1}$, and assume that g_i is completely multiplicative for $i = 1, \ldots, k - 1$. Then, for all n,

$$\begin{aligned}\sum_{d_1|n} \sum_{d_2|d_1} \cdots \sum_{d_k|d_{k-1}} & g_1(d_2)\ldots g_{k-1}(d_k) h_1(d_1/d_2)\ldots h_{k-1}(d_{k-1}/d_k) h_k(d_k) \\ & \cdot f_1(n/d_1)\ldots f_k(n/d_k) \\ &= g_1(n)\ldots g_k(n). \end{aligned} \tag{2}$$

When $k = 1$, (2) is simply the expression of the fact that $g_1 = f_1 * h_1$. We shall complete the proof of (2) by induction on k. Assume $k > 1$, and that (2) holds when k is replaced by $k - 1$. Then, for all n,

$$g_1(n)\cdots g_k(n)$$
$$= g_1(n) \sum_{d_2|n} \cdots \sum_{d_k|d_{k-1}} g_2(d_3)\cdots g_{k-1}(d_k)h_2(d_2/d_3)\cdots h_{k-1}(d_{k-1}/d_k)h_k(d_k) \cdot f_2(n/d_2)\cdots f_k(n/d_k).$$

Since g_1 is completely multiplicative,

$$g_1(n) = g_1(d_2)g_1(n/d_2) = g_1(d_2) \sum_{e|(n/d_2)} h_1(e)f_1(n/d_2e)$$

for every choice of d_2. Hence,

$$g_1(n)\cdots g_k(n)$$
$$= \sum_{d_2|n} \cdots \sum_{d_k|d_{k-1}} g_1(d_2)g_2(d_3)\cdots g_{k-1}(d_k)h_2(d_2/d_3)\cdots h_{k-1}(d_{k-1}/d_k)h_k(d_k) \cdot f_2(n/d_2)\cdots f_k(n/d_k) \sum_{d_2e|n} h_1(e)f_1(n/d_2e).$$

If we write d_1 for d_2e, then d_1 runs over all divisors of n, and for each d_1, d_2 runs over all divisors of d_1, and $e = d_1/d_2$. Hence, we obtain the left-hand side of (2).

Let us look at several examples. If $f_1 = J_k$, $f_2 = J_s$, and $f_3 = J_t$, then, for all n,

$$\sum_{c|n} \sum_{d|c} \sum_{e|d} J_k(n/c)J_s(n/d)J_t(n/e)d^k e^s = n^{k+s+t},$$

the three-function analogue of (1).

If we denote ζ_0 by ζ, so that $\zeta(n) = 1$ for all n, then $\zeta = \mu^{-1}$ and the divisor sum functions σ_k are given by $\sigma_k = \zeta_k * \mu^{-1}$. Thus, for all n,

$$\sum_{d|n} \sum_{e|d} e^k \mu(d/e)\mu(e)\sigma_k(n/d)\sigma_s(n/d) = n^{k+s}.$$

This identity is not included in Stevens' extension of (1). If $\beta(n) =$ the number of integers x such that $1 \leqslant x \leqslant n$ and (x, n) is a square, then $\beta = \zeta_1 * h^{-1}$, where $h(n) = |\mu(n)|$ for all n [3, p. 26]. Hence, for all n,

$$\sum_{d|n} \sum_{e|d} e|\mu(d/e)|\mu(e)\beta(n/d)\sigma(n/e) = n^2.$$

An identity exactly similar to (2) holds in the setting of an arbitrary regular arithmetical convolution. A discussion of these convolutions can be found in W. Narkiewicz's paper [4] or in Chapter 4 of [3]. Let A be a regular arithmetical convolution. An arithmetical function f is called *A-multiplicative* if $f(n) = f(d)f(n/d)$ for all n and all $d \in A(n)$. This generalization of the notion of completely multiplicative function was introduced by K. L. Yocom [6], who obtained several characterizations of such functions.

For $i = 1. \ldots, k$, let $f_i = g_i *_A h_i^{-1}$, where h_i^{-1} is the inverse of h_i with respect to the regular arithmetical convolution A, and assume that g_i is A-multiplicative for $i = 1, \ldots, k-1$. Then, for all n,

$$\sum_{d_1 \in A(n)} \sum_{d_2 \in A(d_1)} \cdots \sum_{d_k \in A(d_{k-1})} g_1(d_2)\ldots g_{k-1}(d_k)h_1(d_1/d_2)\ldots h_{k-1}(d_{k-1}/d_k) \cdot h_k(d_k)f_1(n/d_1)\ldots f_k(n/d_k)$$

$$= g_1(n)\ldots g_k(n). \tag{3}$$

As an example, consider the unitary convolution U, where $d \in U(n)$ means $d|n$ and $(d, n/d) = 1$. Usually, we write $d\|n$ rather than $d \in U(n)$. The unitary Jordan function is $J_k^* = \zeta_k *_U \zeta^{-1}$, where now ζ^{-1} is the inverse of ζ with respect to the unitary convolution. Then, for all n,

$$\sum_{d\|n} \sum_{e\|d} J_k^*(n/d)J_s^*(n/e)e^k = n^{k+s},$$

the unitary analogue of Cohen's identity (1).

The proof of (3) is exactly similar to the proof of (2).

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