ON FIBONACCI AND LUCAS REPRESENTATIONS AND A THEOREM OF LEKKERKERKER

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1. INTRODUCTION

Let $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n = 1, 2, \ldots$, be the Fibonacci numbers and let $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, $n = 0, 1, \ldots$, be the Lucas numbers. According to the Theorem of Zeckendorf (see, for example, [5, p. 74], [6], [1], [8]), every positive integer m has a unique "minimal" representation as a sum of distinct Fibonacci numbers $F_2, F_3, \ldots$ such that no two consecutive Fibonacci numbers are used. If we denote by $f(m)$ the number of Fibonacci numbers in the representation of m, then Lekkerkerker [6] defined the average value

$$\psi_n = \left(\sum_{i=F_{n+1}}^{F_{n+2}-1} f(i)\right) \Big/ F_n$$

and proved that

$$\lim_{n\to\infty} \frac{\psi_n}{n} = \frac{5-\sqrt{5}}{10}. \tag{1}$$

In [7] we gave a very simple proof of (1) and also proved a certain generalization of this result. In order to state this generalization, we introduce some notations and terminology from [7]. Let $1 = a_1 < a_2 < \cdots$ be a strictly increasing sequence of positive integers with the first element equal to 1. We call this an *A-sequence*. Suppose that m is a positive integer. We write

$$m = a_{(1)} + a_{(2)} + \cdots + a_{(s)}, \tag{2}$$

where $a_{(1)}$ is the greatest element of the A-sequence $\leqslant m$, $a_{(2)}$ is the greatest element of the A-sequence $\leqslant m - a_{(1)}$, and, generally, $a_{(i)}$ is the greatest element of the A-sequence $\leqslant m - a_{(1)} - a_{(2)} - \cdots - a_{(i-1)}$. We denote by $h(m)$ the number s in (2), that is, the number of terms in the representation of m.

Suppose that $k \geqslant 2$ is a positive integer and define an A-sequence by $a_1 = 1$, $a_2 = k$, and $a_{n+2} = a_{n+1} + a_n$, $n = 1, 2, \ldots$. We call this a *recursive A=sequence*. If $a_2 = k = 3$, then $a_n = L_n$, $n = 1, 2, \ldots$. If $a_2 = k = 2$, then

$a_n = F_{n+1}$, $n = 1, 2, \ldots$, and (2) is the Zeckendorf representation [7, Lemma 5.12, p. 45], so that $h(m) = f(m)$.

Consider now a recursive A-sequence. If $a_2 = k$, we defined

$$\psi_k(n) = \left(\sum_{i=a_n}^{a_{n+1}-1} h(i)\right)\Big/ a_{n-1},$$

so that $\psi_2(n) = \psi_n$, and proved [7, Theorem 5.15, p. 46] that for all $a_2 = k \geqslant 2$,

$$\lim_{n\to\infty} \frac{\psi_k(n)}{n} = \frac{5 - \sqrt{5}}{10} \tag{3}$$

a generalization of (1).

In [4] Daykin has given a different generalization of (1). If h and k are positive integers such that $h \leqslant k \leqslant h + 1$, then he defined [4, p. 144] the $(h, k)^{\text{th}}$ *Fibonacci sequence* (v_i) in the following way:

$$\begin{aligned} v_i &= i && \text{for } 1 \leqslant i \leqslant k,\\ v_i &= v_{i-1} + v_{i-h} && \text{for } k < i < h + k,\\ v_i &= v_{i-1} + v_{i-k} + (k - h) && \text{for } i \geqslant h + k. \end{aligned} \tag{4}$$

Clearly, the Fibonacci numbers F_2, F_3, ... are given by the $(2, 2)^{\text{th}}$ Fibonacci sequence.

Daykin generalizes the Theorem of Zeckendorf by proving [4, Theorem C, p. 144] that, if (v_i) is the $(h, k)^{\text{th}}$ Fibonacci sequence, then for each positive integer m there is one, and only one, system of positive integers $i_1, i_2, \ldots, i_d$ such that

$$m = v_{i_1} + v_{i_2} + \cdots + v_{i_d}, \tag{5}$$

where $i_2 \geqslant i_1 + h$ if $d > 1$, and $i_{\nu+1} \geqslant i_\nu + k$ for $2 \leqslant \nu < d$. [We note that the $(h, k)^{\text{th}}$ Fibonacci sequence is an A-sequence, and it is easy to see that the representation (5) is the same as (2).]

Let ψ_n denote the average number of summands required in (5) for all those positive integers m such that $v_n \leqslant m < v_{n+1}$. Then [4, Theorem E, p. 144] for $k \geqslant 2$,

$$\lim_{n\to\infty} \frac{\psi_n}{n} = \frac{\theta - 1}{1 + k(\theta - 1)}, \tag{6}$$

where $\theta = \theta(k)$ is the positive real solution of the equation $z - 1 = z^{1-k}$.

In this paper we consider three other kinds of unique representations using Fibonacci and Lucas numbers and prove the following corresponding results. Let $f'(m)$ denote the number of elements in the "maximal" representation (see [5, p. 74], [2]) of m using Fibonacci numbers F_2, F_3, ... (where no "gaps" formed by two consecutive Fibonacci numbers are allowed). Let $g(m)$ denote the number of elements in the "minimal" representation (see [5, p. 76], [3], [8]) of m and let $g'(m)$ denote the number of elements in the "maximal" representation (see [5, p. 77], [3]) of m using Lucas numbers. These are similar to the corresponding Fibonacci representations but with certain additional restrictions to ensure uniqueness. We define

$$\psi_n' = \left(\sum_{i=F_{n+1}}^{F_{n+2}-1} f'(i)\right)\Big/F_n, \quad \lambda_n = \left(\sum_{i=L_{n+1}}^{L_{n+2}-1} g(i)\right)\Big/L_n,$$

and
$$\lambda_n' = \left(\sum_{i=L_{n+1}}^{L_{n+2}-1} g'(i)\right)\Big/L_n.$$

Then we have

$$\lim_{n\to\infty}\frac{\lambda_n}{n} = \frac{5-\sqrt{5}}{10}, \tag{7}$$

$$\lim_{n\to\infty}\frac{\lambda_n'}{n} = \frac{5+\sqrt{5}}{10}, \tag{8}$$

and

$$\lim_{n\to\infty}\frac{\psi_n'}{n} = \frac{5+\sqrt{5}}{10}. \tag{9}$$

2. "MINIMAL" LUCAS REPRESENTATIONS

Let $a_1 = 1 = L_1$, $a_2 = L_0$, and $a_n = L_{n-1}$, $n = 3, 4, \ldots$, so that

$$a_{n+2} = a_{n+1} + a_n,\ n = 3, 4, \ldots . \tag{10}$$

Lemma 1: The representation of a positive integer m corresponding to this A-sequence is the "minimal" Lucas representation.

Proof: Similar to that of Lemma 5.12 in [7, p. 45]. ∎

It follows that $g(m) = h(m)$ for every positive integer m. Let

$$S(n) = \sum_{i=1}^{n} h(i) \quad \text{and} \quad S'(n) = S(a_{n+1} - 1) \quad [7, \text{p. } 7].$$

Then it follows from (10) that we have (compare with Theorem 5.4 in [7, p. 41])

$$S'(n+2) = S'(n+1) + S'(n) + L_n,\ n = 2, 3, \ldots . \tag{11}$$

Lemma 2: $S'(n) = n \cdot F_{n-1}$, $n = 2, 3, \ldots$.

Proof: Easily by induction, using (11) and [5, (I_8), p. 56]. ∎

It follows that

$$\sum_{i=L_{n+1}}^{L_{n+2}-1} g(i) = \sum_{i=a_{n+2}}^{a_{n+3}-1} h(i) = S'(n+2) - S'(n+1) \tag{12}$$
$$= (n+2)F_{n+1} - (n+1)F_n = n \cdot F_{n-1} + L_n,\ n = 2, 3, \ldots .$$

(This holds also for $n = 1$, if we define $F_0 = 0$ as usual.) From (12), it follows that (7) holds, because

$$\lim_{n\to\infty} \frac{F_{n-1}}{L_n} = \frac{5-\sqrt{5}}{10} \quad \text{(see, for example, [7, (5.36), p. 47]).} \tag{13}$$

3. "MAXIMAL" LUCAS REPRESENTATIONS

Lemma 3: Suppose that m is a positive integer such that $L_{n+1} \leqslant m \leqslant L_{n+2} - 1$, where $n \geqslant 1$. Then the greatest-indexed Lucas number in the "maximal" Lucas representation of m is L_n.

Proof: This follows from Theorem 2 in [3, p. 250]. ■

Let

$$a(n) = \sum_{i=L_{n+1}}^{L_{n+2}-1} g'(i),\ n \geqslant 0,$$

so that $\lambda'_n = a(n)/L_n$.

Lemma 4: $a(n) = a(n-1) + a(n-2) + L_n,\ n = 2, 3, \ldots .$

Proof: Since $a(0) = a(1) = 2$, $a(2) = 7$, and $L_2 = 3$, the equation clearly holds for $n = 2$. Let $L_{n+1} \leqslant m \leqslant L_{n+2} - 1$, where $n \geqslant 3$. According to Lemma 3, the greatest-indexed Lucas number in the representation of m is L_n. Let $m' = m - L_n$. Then

$$L_{n-1} \leqslant m' \leqslant L_{n+1} - 1. \tag{14}$$

According to Lemma 3, if $L_n \leqslant m' \leqslant L_{n+1} - 1$, then the greatest-indexed Lucas number in the representation of m' is L_{n-1}, and if $L_{n-1} \leqslant m' \leqslant L_n - 1$, then it is L_{n-2}. It follows that in both cases we get the representation of m by adding L_n to the representation of m'. It follows that

$$g'(m) = g'(m') + 1, \tag{15}$$

which, together with (14), clearly completes the proof. ■

Lemma 5: $a(n) = n \cdot F_{n+1} + L_n,\ n = 0, 1, \ldots .$

Proof: Easily by induction using Lemma 4. ■

It follows that (8) holds because

$$\lim_{n\to\infty}\frac{F_{n+1}}{L_n} = \frac{5+\sqrt{5}}{10} \quad \text{(see, for example, [7, (5.34), p. 47]).} \tag{16}$$

4. "MAXIMAL" FIBONACCI REPRESENTATIONS

Let

$$b(n) = \sum_{i=F_{n+1}}^{F_{n+2}-1} f'(i),\ n \geqslant 1,$$

so that $\psi_n' = b(n)/F_n$. Let

$$c(n) = \sum_{i=F_{n+1}-1}^{F_{n+2}-2} f'(i),\ n \geqslant 2.$$

Then we have

Lemma 6: $b(n) = c(n) + \dfrac{1+(-1)^{n+1}}{2}$, $n = 2, 3, \ldots$.

Proof: $b(n) - c(n) = f'(F_{n+2} - 1) - f'(F_{n+1} - 1)$. We use the formulas (see, for example, [5, (I_5), (I_6), p. 56])

$$F_{2k} - 1 = F_{2k-1} + F_{2k-3} + \cdots + F_5 + F_3,\ k = 2, 3, \ldots, \tag{17}$$

and

$$F_{2k+1} - 1 = F_{2k} + F_{2k-2} + \cdots + F_4 + F_2,\ k = 1, 2, \ldots . \tag{18}$$

If n is even, $n = 2k$, we get $f'(F_{n+2} - 1) - f'(F_{n+1} - 1) = k - k = 0$ and if n is odd, $n = 2k + 1$, we get $f'(F_{n+2} - 1) - f'(F_{n+1} - 1) = (k + 1) - k = 1$. ∎

Lemma 7: Let $F_{n+1} - 1 \leqslant m \leqslant F_{n+2} - 2$, where $n \geqslant 2$. Then the greatest Fibonacci number in the "maximal" representation of m is F_n.

Proof: [2, Theorem 1, p. 2]. ∎

In a similar fashion as in the case of "maximal" Lucas representations, it follows now that

$$c(n) = c(n-1) + c(n-2) + F_n,\ n = 4, 5, \ldots . \tag{19}$$

Lemma 8: $c(n) = (1/5)(n \cdot L_{n+1} - 3F_n)$, $n = 2, 3, \ldots$.

Proof: Easily by induction using (19) and [5, (I_9), p. 56]. ∎

Formula (9) now follows from Lemma 8 and Lemma 6 using the fact that

$$\lim_{n\to\infty}\frac{L_{n+1}}{F_n} = \frac{5+\sqrt{5}}{2} \quad \text{(see, for example, [7, (5.36), p. 47]).} \tag{20}$$

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