# A NOTE ON FIBONACCI TREES AND THE ZECKENDORF REPRESENTATION OF INTEGERS

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The *Fibonacci numbers* are defined, as usual, by the recurrence

$$F_0 = 0,\ F_1 = 1,\ F_k = F_{k-1} + F_{k-2},\ k > 1.$$

The *Fibonacci tree of order k*, denoted $T_k$, can be constructed inductively as follows: If $k = 0$ or $k = 1$, the tree is simply the root 0. If $k > 1$, the root is $F_k$; the left subtree is $T_{k-1}$; and the right subtree is $T_{k-2}$ with all node numbers increased by $F_k$. $T_6$ is shown in Figure 1. For an elegant role of the node numbers in the Fibonacci search algorithm, the reader is referred to [5].

Fibonacci trees have been studied in detail by Horibe [2], [3]. The aim of this note is to present some additional considerations on Fibonacci tree codes and to explore the relationships existing between the codes and the *Zeckendorf representation* of integers.

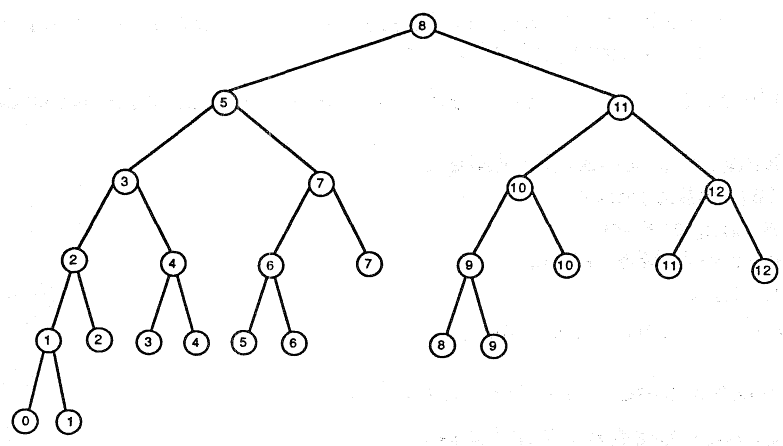


Figure 1. The Fibonacci Tree of Order 6, $T_6$

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Recall that each integer $N$, $0 \leqslant N < F_{k+1}$, has the following unique Zeckendorf representation in terms of Fibonacci numbers [6]:

$$N = \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k, \text{ where } \alpha_i \in \{0, 1\} \text{ and } \alpha_i \alpha_{i-1} = 0.$$

Let us write this as $\alpha_k \alpha_{k-1} \alpha_{k-2} \ldots \alpha_3 \alpha_2$. The Zeckendorf representation of an integer then provides a binary sequence, called a *Fibonacci sequence*, that does not contain two consecutive ones, and the number of Fibonacci sequences of length $k - 1$ is exactly $F_{k+1}$.

The Zeckendorf representation of integers perserves the lexicographic ordering based on $0 < 1$ (see [1]).

A *tree code* is the code obtained by labeling each branch of a tree with a code symbol and representing each terminal node with the path of labels from the root to it. We stress that tree codes are *prefix* codes (i.e., no codeword is the beginning of any other codeword) and have a natural encoding and decoding. Moreover, tree codes preserve the order structure of the encoded set in the sense that, if $x$ precedes $y$, the codeword for $x$ *lexicographically* precedes the codeword for $y$.

In the sequel, we use 0 for each left branch and 1 for each right branch in a binary tree. The Fibonacci code, denoted $C_k$, is the binary code obtained in this way from $T_k$. For example, $C_6$ is shown in the following table.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 00000 | 5 | 0100 | 10 | 101 |
| 1 | 00001 | 6 | 0101 | 11 | 110 |
| 2 | 0001 | 7 | 011 | 12 | 111 |
| 3 | 0010 | 8 | 1000 | | |
| 4 | 0011 | 9 | 1001 | | |

The first result of this note is the determination of the asymptotic proportions of zeros and ones in the Fibonacci codes.

Let $N_k^0$ and $N_k^1$ denote the total number of 0's and 1's in $C_k$, respectively, and let $N_k = N_k^0 + N_k^1$ denote the total number of symbols. For example, $N_6^0 = 30$ and $N_6^1 = 20$. Put $p = \lim\limits_{k \to \infty}(N_k^0/N_k)$ and $q = 1 - p = \lim\limits_{k \to \infty}(N_k^1/N_k)$. We will show the following

**Theorem 1:** $p = \dfrac{1}{\Phi}$ and $q = 1 - \dfrac{1}{\Phi}$, where $\Phi = \lim\limits_{n \to \infty} \dfrac{F_{n+1}}{F_n}$ is the *golden ratio* $\dfrac{1 + \sqrt{5}}{2}$.

**Proof:** From the inductive construction of the Fibonacci tree and the fact that $T_k$ has $F_{k+1}$ terminal nodes, one has the following equations:

$$N_k = F_{k+1} + N_{k-1} + N_{k-2};$$

$$N_k^0 = F_k + N_{k-1}^0 + N_{k-2}^0;$$
$$N_k^1 = F_{k-1} + N_{k-1}^1 + N_{k-2}^1.$$

These equations, applied recursively, give

$$N_k = \sum_{i=0}^{k-1} F_i F_{k-i+2}, \quad N_k^0 = \sum_{i=0}^{k-1} F_i F_{k-i+1}, \quad N_k^1 = \sum_{i=0}^{k-1} F_i F_{k-i}.$$

Therefrom one gets: $N_k^0/N_k = \sum_{i=0}^{k-1} F_i F_{k-i+1} \Big/ \sum_{i=0}^{k-1} F_i F_{k-i+2}$.

To evaluate the asymptotic behavior of $\sum_{i=0}^{k-1} F_i F_{k-i+j}$, we use Binet's formula

$$F_k = \frac{1}{\sqrt{5}}(\Phi^k - \Gamma^k), \text{ where } \Gamma = \frac{1-\sqrt{5}}{2}.$$

We then have

$$\begin{aligned}\sum_{i=0}^{k-1} F_i F_{k-i+j} &= \frac{1}{5}\left(\sum_{i=0}^{k-1} \Phi^{k+j} + \sum_{i=0}^{k-1} \Gamma^{k+j} - \sum_{i=0}^{k-1} \Phi^i \Gamma^{k-i+j} - \sum_{i=0}^{k-1} \Phi^{k-i+j}\Gamma^i\right)\\
&= \frac{1}{5}\left(k\Phi^{k+j} + k\Gamma^{k+j} - \Gamma^{j+1}\frac{\Gamma^k - \Phi^k}{\Gamma - \Phi} - \Phi^{j+1}\frac{\Phi^k - \Gamma^k}{\Phi - \Gamma}\right)\\
&= \frac{1}{5}\left(k\Phi^{k+j} + k\Gamma^{k+j} - \frac{\Gamma^{j+1}}{\sqrt{5}}(\Phi^k - \Gamma^k) - \frac{\Phi^{j+1}}{\sqrt{5}}(\Phi^k - \Gamma^k)\right)\\
&= \frac{1}{5}k\Phi^{k+j} + O(\Phi^k).\end{aligned}$$

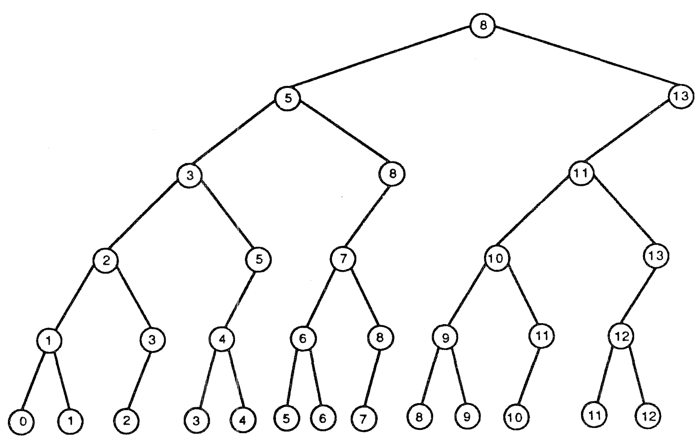


Figure 2. The Uniform Fibonacci Tree of Order 6, $U_6$

From the above, one finally obtains

$$\lim_{k\to\infty} \frac{N_k^0}{N_k} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} F_i F_{k-i+1}}{\sum_{i=0}^{k-1} F_i F_{k-i+2}} = \frac{1}{\Phi}. \blacksquare$$

The remainder of this note is devoted to exploring relationships between the Fibonacci codes and the Zeckendorf representation of integers. In particular, we show that the Zeckendorf representation of integers can be obtained as a variant of the Fibonacci codes by inserting some redundant digits 0.

To this end, let us define the *uniform Fibonacci tree* of order $k$ (denoted $U_k$) as follows: For $k \leqslant 2$, the uniform Fibonacci tree coincides with the Fibonacci tree. If $k > 2$, the root is $F_k$; the left subtree is $U_{k-1}$; the right subtree has root $F_k + F_{k-1}$ whose right subtree is empty and whose left subtree is $U_{k-2}$ with all numbers increased by $F_k$.

A uniform Fibonacci tree is the Fibonacci tree with dummy nodes after each right branch that force the leaves to be at the same level. The uniform Fibonacci tree can be obtained from the branch labeling of the Fibonacci tree, as described in [3]. The relationships between this labeling and the Zeckendorf representation of integers have been unnoticed. Figure 2 above shows $U_6$. Some properties of $U_k$ are given in the following theorems.

**Theorem 2:** $U_k$ has $F_{i+2}$ nodes at level $i$, $0 \leqslant i \leqslant k-1$.

**Proof:** Theorem 2 is trivially true for $k = 1, 2$. Suppose it is true for each $U_i$, $i < k$ $(k > 2)$. We prove that it is true for $U_k$.

Let us denote by $L(i, k)$ the number of nodes that $U_k$ has at level $i$. The construction of $U_k$ implies

$$L(0, k) = F_2, \quad L(1, k) = F_3,$$

and

$$L(i, k) = L(i-1, k-1) + L(i-2, k-2), \; 2 \leqslant i \leqslant k-1.$$

By the induction hypothesis, this gives $L(i, k) = F_{i+1} + F_i = F_{i+2}$. ■

**Corollary 1:** $U_k$ is obtained by adding $F_k - 1$ internal nodes to $T_k$.

**Proof:** From Theorem 2, $U_k$ has $\sum_{i=2}^{k} F_i = F_{k+2} - 2$ internal nodes. Since $T_k$ has $F_{k+1} - 1$ internal nodes, we get that $U_k$ has $F_{k+2} - 2 - F_{k+1} + 1 = F_k - 1$ additional nodes. ■

Similarly, as was done in [3] for Fibonacci trees, it is possible to classify terminal nodes of $U_k$ into:

$\mathcal{R}$-nodes, the terminal nodes that are right sons, and

$\mathcal{L}$-nodes, the terminal nodes that are left sons.

**Lemma 1:** $U_k$ has $F_{k-1}$ $\mathcal{R}$-nodes and $F_k$ $\mathcal{L}$-nodes.

**Proof:** By induction. Trivially true for $k = 2, 3$. Suppose the lemma is true for each uniform Fibonacci tree of order less than $k$, $k > 3$. The definition of $\mathcal{R}$-nodes and $\mathcal{L}$-nodes implies that the type ($\mathcal{R}$ or $\mathcal{L}$) determination within each of the left and right subtrees of any uniform Fibonacci tree gives the correct type determination in the whole tree. Hence, by the construction of $U_k$ and by the induction hypothesis, $U_k$ has $F_{k-2} + F_{k-3}$ $\mathcal{R}$-nodes and $F_{k-1} + F_{k-2}$ $\mathcal{L}$-nodes. This completes the proof. ∎

As was done in [2] for Fibonacci trees, and as Theorem 2 suggests, one can construct $U_{k+1}$ by properly splitting terminal nodes of $U_k$. However, the recursive construction for uniform Fibonacci trees is slightly different from that described in [2] for Fibonacci trees. This time, all terminal nodes generate offsprings.

**Theorem 3:** If each $\mathcal{R}$-node of $U_k$, $k \geqslant 2$, generates only the left node and each $\mathcal{L}$-node generates two nodes, then the resulting tree that has $F_k$ $\mathcal{R}$-nodes and $F_{k-1} + F_k$ $\mathcal{L}$-nodes is exactly $U_{k+1}$.

**Proof:** By induction. Suppose the theorem is true for each $U_i$, $i < k$, $k > 3$ (when $k = 2, 3$, the assertion is easily shown). $U_k$ has, as its left subtree, $U_{k-1}$ with $F_{k-2}$ $\mathcal{R}$-nodes and $F_{k-1}$ $\mathcal{L}$-nodes. Making terminal nodes of this $U_{k-1}$ generate offsprings produces $U_k$ by the induction hypothesis. Similarly, the right subtree of $U_k$ has empty right subtree and has $U_{k-2}$ as the left subtree. Making the $F_{k-3}$ $\mathcal{R}$-nodes and the $F_{k-2}$ $\mathcal{L}$-nodes of this $U_{k-2}$ generate offsprings produces $U_{k-1}$ by the induction hypothesis. Therefore, making all $\mathcal{R}$-nodes of $U_k$ generate left sons and all $\mathcal{L}$-nodes generate two sons produces $U_{k+1}$. ∎

We now relate the tree code of $U_k$, *the uniform Fibonacci tree code* of order $k$ (denoted in the sequel by $B_k$), to the Zeckendorf representation of integers. For example, $B_6$ is given by:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 00000 | 5 | 01000 | 10 | 10010 |
| 1 | 00001 | 6 | 01001 | 11 | 10100 |
| 2 | 00010 | 7 | 01010 | 12 | 10101 |
| 3 | 00100 | 8 | 10000 | | |
| 4 | 00101 | 9 | 10001 | | |

**Lemma 2:** The uniform Fibonacci code of order $k$ is the set of all Fibonacci sequences of length $k - 1$.

**Proof:** From the construction of the uniform Fibonacci tree, the uniform Fibonacci code does not allow two consecutive 1's in any codeword and contains $F_{k+1}$ distinct codewords of length $k - 1$. The number of Fibonacci sequences of length $k - 1$ is also given by $F_{k+1}$. ■

**Theorem 4:** In a uniform Fibonacci code, the codeword that represents the terminal node $i$ is the Zeckendorf representation of the integer $i$.

**Proof:** From Lemma 2, the uniform Fibonacci tree code of order $k$ is the set of Fibonacci sequences of length $k - 1$. By definition, they provide the Zeckendorf representation of nonnegative integers $< F_{k+1}$. Since the Zeckendorf representation preserves the lexicographic ordering, the assertion is a straightforward consequence of the order-preserving property of tree codes. ■

Uniform Fibonacci trees, therefore, provide an efficient pretty mechanism for obtaining the Zeckendorf representation of integers. The procedure is:

*Given the integer $i$, $0 \leqslant i < F_{k+1}$, construct the uniform Fibonacci tree of order $k$. The Zeckendorf representation of $i$ is the path of labels from the root to terminal node $i$.*

It is also worthwhile to note that the uniform Fibonacci trees in the setting of the Fibonacci numeration system play a role analogous to that of the complete binary trees in the setting of the binary numeration system:

*The number of nodes at each level is given by a Fibonacci number (power of 2, in the binary case);*

*The path of labels to a terminal node is the Zeckendorf representation (the binary representation, in the binary case).*

The last result is the determination of the number $\overline{N}_k^1$ of 1's and the number $\overline{N}_k^0$ of 0's in $B_k$. With the same notation of Theorem 1, we have

**Theorem 5:** $\overline{N}_k^1 = N_k^1$; $\overline{N}_k^0 = N_k^0 + N_k^1 - F_{k-1}$, $k \geqslant 2$.

**Proof:** The first part is immediate from the construction of trees $T_k$ and $U_k$. The second part can be proved by induction. Suppose Theorem 5 is true for each uniform Fibonacci tree of order less than $k$, $k > 3$ (when $k = 2, 3$, the assertion is trivially true). By the construction of $U_k$, one has the equation:

$$\overline{N}_k^0 = (F_k + \overline{N}_{k-1}^0) + (F_{k-1} + \overline{N}_{k-2}^0).$$

By the induction hypothesis, this gives

$$\overline{N}_k^0 = F_k + F_{k-1} + N_{k-1}^0 + N_{k-1}^1 - F_{k-2} + N_{k-2}^0 + N_{k-2}^1 - F_{k-3}.$$

Since $N_k^0 = F_k + N_{k-1}^0 + N_{k-2}^0$ and $N_k^1 = F_{k-1} + N_{k-1}^1 + N_{k-2}^1$ (see Theorem 1), the assertion is true. ■

Theorem 5 allows immediate computation of the asymptotic proportion of 1's (and 0's) in Fibonacci sequences (see [4]). Indeed, denoting by $p$, $q$ and $\overline{p}$, $\overline{q}$, respectively, the asymptotic proportions of 0's and 1's in $C_k$ and $B_k$, and recalling Theorem 1 and its proof, one obtains

$$\overline{q} = 1 - \overline{p} = \lim_{k\to\infty} \frac{\overline{N}_k^1}{\overline{N}_k} = \lim_{k\to\infty} \frac{N_k^1}{N_k + N_k^1 - F_{k-1}} = \frac{q}{1+q} = \frac{\Phi - 1}{2\Phi - 1} = \frac{5 - \sqrt{5}}{10}.$$

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