SOME OBSERVATIONS ON THE CLASSICAL CUBOID AND ITS PARAMETRIC SOLUTIONS

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1. The classical cuboid has integral edges and face diagonals. We require integer solutions of the Diophantine equations:

$$x^2 + y^2 = u^2, \quad x^2 + z^2 = v^2, \quad \text{and} \quad y^2 + z^2 = \omega^2. \tag{1.1}$$

The first parametric solution was given by Saunderson (Dickson [1], p. 497) and subsequent two-parameter solutions have been given by a number of writers; a listing of these authors can be found in Kraitchik [2]. The general solution of equations (1.1) is unknown. In this paper a method is given which leads to an infinity of two-parameter solutions which are of ever-increasing degree and complexity.

2. A solution of (1.1) is given by

$$\begin{aligned} x &= (a^2 - d^2)(c^2 - b^2) \\ y &= 2ad(c^2 - b^2) \\ z^2 &= 4c^2b^2\left(a^2 + \frac{2abd}{c} - d^2\right)\left(a^2 + \frac{2acd}{b} - d^2\right), \end{aligned} \tag{2.1}$$

because

$$\begin{aligned} x^2 + y^2 &= ((c^2 - b^2)(a^2 + d^2))^2 \\ x^2 + z^2 &= ((a^2 - d^2)(b^2 + c^2) + 4abcd)^2 \\ y^2 + z^2 &= 4(ad(b^2 + c^2) + bc(a^2 - d^2))^2. \end{aligned}$$

We see from these equations that a cuboid with two integral edges and integral face diagonals has a four-parameter solution. The problem here is to make z rational.

Putting $a/d = w$ and $b/c = D$ (say), where w and D are rationals, we have

$$z^2 = 4c^2b^2d^4(w^2 + 2Dw - 1)\left(w^2 + \frac{2}{D}w - 1\right). \tag{2.2}$$

If we multiply the quadratics and put $A = D + 1/D$, we require rational solutions of

$$w^4 + 2Aw^3 + 2w^2 - 2Aw + 1 = t^2. \tag{2.3}$$

We wish to determine solutions of (2.3) in the form $w = w(A)$. If (2.3) has a rational solution $w = w_0$, then it also has a rational solution

$$w = -\frac{1}{w_0}.$$

But this will just interchange a and d and will not effect the solution.

We can equate (2.3) to the square of a quadratic in w in the usual way, to show that there is a rational solution

$$w = \frac{A}{4}, \quad t = \frac{3A^2}{16} - 1. \tag{2.4}$$

This gives the classical solution of Saunderson:

$$\begin{aligned} x &= (c^2 - b^2)((b^2 + c^2)^2 - 16b^2c^2) \\ y &= 8bc(c^4 - b^4) \\ z &= 2bc(3(b^2 + c^2)^2 - 16b^2c^2). \end{aligned} \tag{2.5}$$

Equation (2.2) has another simple solution. Putting $w = 1/2D$, we see that $w^2 + 2Dw - 1$ is square, and we require

$$\frac{5}{4D^2} - 1 = \square.$$

This has the standard rational solution

$$D = \frac{\alpha^2 + \alpha\beta - \beta^2}{\alpha^2 + \beta^2} \quad \text{and} \quad \square = \left(\frac{\alpha^2 - 4\alpha\beta - \beta^2}{2(\alpha^2 + \alpha\beta - \beta^2)}\right)^2,$$

which gives

$$a = \alpha^2 + \beta^2, \quad b = \alpha^2 + \alpha\beta - \beta^2, \quad c = \alpha^2 + \beta^2, \quad d = 2(\alpha^2 + \alpha\beta - \beta^2),$$

and we have the solution:

$$\begin{aligned} x &= \alpha\beta(\alpha^2 - \beta^2)(3\alpha - \beta)(3\beta + \alpha)(2\alpha + \beta)(2\beta - \alpha) \\ y &= 4\alpha\beta(\alpha^2 + \beta^2)(2\alpha + \beta)(2\beta - \alpha)(\alpha^2 + \alpha\beta - \beta^2) \\ z &= 2(\alpha^2 + \beta^2)^2(\alpha^2 + \alpha\beta - \beta^2)(\alpha^2 - 4\alpha\beta - \beta^2). \end{aligned} \tag{2.6}$$

3. To determine further solutions of (2.3), we can put $w = n + w_0$, where $w_0^4 + 2Aw_0^3 + 2w_0^2 - 2Aw_0 + 1 = t_0^2$, and write

$$\begin{aligned} &n^4 + (4w_0 + 2A)n^3 + (6w_0^2 + 6Aw_0 + 2)n^2 + (4w_0^3 + 6Aw_0^2 + 4w_0 - 2A)n + t_0^2 \\ &= (Cn^2 + Bn + t_0)^2 \quad \text{(say)}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2Bt_0 &= 4w_0^3 + 6Aw_0^2 + 4w_0 - 2A \\ B^2 + 2Ct_0 &= 6w_0^2 + 6Aw_0 + 2 \end{aligned}$$

and

$$w = \frac{2BC - 4w_0 - 2A}{1 - C^2} + w_0.$$

These equations give:

$$w = \frac{Aw_0^9 + 12w_0^8 + 12Aw_0^7 + 32w_0^6 + 30Aw_0^5 + 24w_0^4 - 36Aw_0^3 + 9Aw_0 - 4}{4w_0^9 + 9Aw_0^8 - 36Aw_0^6 - 24w_0^5 + 30Aw_0^4 - 32w_0^3 + 12Aw_0^2 - 12w_0 + A}. \tag{3.1}$$

If we put $w_0 = A/4$, then the next solution generated is

$$w = \frac{A^{10} + 240A^8 + 9728A^6 - 122880A^4 + 589824A^2 - 1048576}{8A(5A^8 - 288A^6 + 3072A^4 + 8192A^2 - 65536)}.$$

Putting $D = 2 = b/c$, we obtain $A = 5/2$ and $w = 602697401/880248720$. Hence, we have a cuboid with $b = 2$, $c = 1$, $a = 602697401$, and $d = 880248720$.

Equation (3.1) will generate an infinity of rational solutions w, and each such solution gives a two-parameter solution of equations (1.1). It is evident that these solutions increase very rapidly in degree and complexity. The solutions do not necessarily give independent parametric formulas. If we put $w_0 = 1$, then $w = \frac{A+4}{A-4}$, which, again, gives Saunderson's solution (2.5).

4. It is seen that the solution

$$w = \frac{A}{4} = \frac{1}{4}\left(D + \frac{1}{D}\right)$$

makes both quadratics, $w^2 + 2Dw - 1$ and $w^2 + \frac{2}{D}w - 1$, simultaneously square. We will now consider this further.

We have

$$w^2 + 2Dw - 1 = \left(\frac{\alpha^2 + 2D\alpha - 1}{2\alpha + 2D}\right)^2 \quad \text{if } w = \frac{\alpha^2 + 1}{2\alpha + 2D} \tag{4.1}$$

and

$$w^2 + \frac{2}{D}w - 1 = \left(\frac{\beta^2 + \frac{2}{D}\beta - 1}{2\beta + \frac{2}{D}}\right)^2 \quad \text{if } w = \frac{\beta^2 + 1}{2\beta + \frac{2}{D}}, \tag{4.2}$$

where α and β are arbitrary rationals such that w is finite. Equating (4.1) and (4.2), we require rationals α and β such that

$$\frac{\alpha^2 + 1}{2\alpha + 2D} = \frac{\beta^2 + 1}{2\beta + \frac{2}{D}}. \tag{4.3}$$

If $\alpha = \beta$, then $D = 1$, which is trivial. If $\alpha = -\beta$, then we again obtain the classical solution (2.5). Thus, we have

$$(\alpha + D)(\beta^2 + 1) = \left(\beta + \frac{1}{D}\right)(\alpha^2 + 1).$$

Put $\alpha + D = K\left(\beta + \frac{1}{D}\right)$ and $\beta^2 + 1 = \frac{1}{K}(\alpha^2 + 1)$ for some rational K:

$$\therefore \quad \beta^2(K^2 - K) + \beta\left(\frac{2K^2}{D} - 2KD\right) + \left(\frac{K^2}{D^2} - 3K + D^2 + 1\right) = 0$$

$$\therefore \quad \beta = \left(KD - \frac{K^2}{D} \pm \left(\frac{(1 + D^2)K^3}{D^2} - 4K^2 + (1 + D^2)K\right)^{1/2}\right)\Big/ K^2 - K.$$

We require

$$\frac{(1 + D^2)}{D^2}K^3 - 4K^2 + (1 + D^2)K = \square. \tag{4.4}$$

Multiply equation (4.4) by $\left(\frac{D^2 + 1}{D^2}\right)^2$ and put $\frac{(D^2 + 1)K}{D^2} = m$ (say).

$$\therefore \quad m^3 - 4m^2 + \left(\frac{D^2 + 1}{D}\right)^2 m = \square.$$

Let us put, as before, $A = D + 1/D$, then we have

$$m^3 - 4m^2 + A^2 m = t^2. \tag{4.5}$$

Equation (4.5) is an elliptic curve and has the obvious rational solution $m = 4$. We can see, by direct substitution, that if $m = m_0$ is a rational solution then $m = A^2/m_0$ is also a rational solution. Employing the same technique as before, we can put $m = n + m_0$ and consider

$$n^3 + n^2(3m_0^2 - 4) + n(3m_0 - 8m_0 + A^2) + t_0^2 = (Bn + t_0)^2, \tag{4.6}$$

which gives

$$m = \frac{(m_0^2 - A^2)^2}{4(m_0^3 - 4m_0^2 + A^2 m_0)}. \tag{4.7}$$

The right-hand side of (4.7) is unchanged if m_0 is replaced by A^2/m_0. We can therefore generate two sequences of solutions starting with $m_0 = 4$. Thus, we have

$$m_0 = 4 \quad \text{and} \quad \frac{A^2}{4}$$

$$m_1 = \frac{(16 - A^2)^2}{16A^2} \quad \text{and} \quad \frac{16A^4}{(16 - A^2)^2}$$

$$m_2 = \frac{((16 - A^2)^4 - 256A^6)^2}{64A^2(A^2 - 16)^2(A^4 + 64A^2 - 256)} \quad \text{and} \quad \frac{64A^4(A^2 - 16)^2(A^4 + 64A^2 - 256)}{((16 - A^2)^4 - 256A^6)^2}$$

etc.

Using these values of m we can determine β, and hence α, as a rational function of D. This will then give w as a rational function of D and will lead to a two-parameter solution. For $m = 4$, we have solution (2.5). For $m = A^2/4$, we have

$$\alpha = \frac{D^4 + 8D^2 - 1}{2D(D^2 - 3)} \quad \text{and} \quad \beta = \frac{5D^2 + 1}{D(D^2 - 3)}$$

with $\quad w = \dfrac{(D^2 + 1)(D^4 + 18D^2 + 1)}{4D(3D^4 - 10D^2 + 3)}.$

With this values for w, we have

$$w^2 + 2Dw - 1 = \left(\frac{5D^6 + 27D^4 - 41D^2 + 1}{4D(3D^4 - 10D^2 + 3)}\right)^2$$

and

$$w^2 + \frac{2}{D}w - 1 = \left(\frac{D^6 - 41D^4 + 27D^2 + 5}{4D(3D^4 - 10D^2 + 3)}\right)^2.$$

Putting $D = b/c$ and removing common factors gives the solution:

$$\begin{aligned} x &= (c^2 - b^2)((b^2 + c^2)^2(b^4 + 18b^2c^2 + c^4)^2 \\ &\qquad - 16b^2c^2(3b^4 - 10b^2c^2 + 3c^4)^2) \end{aligned} \tag{4.8}$$

$$y = 8bc(c^4 - b^4)(b^4 + 18b^2c^2 + c^4)(3b^4 - 10b^2c^2 + 3c^4)$$

$$z = 2bc(b^6 - 41b^4c^2 + 27b^2c^4 + 5c^6)(5b^6 + 27b^4c^2 - 41b^2c^4 + c^6)$$

Putting $b = 2$, $c = 1$ gives

$$x = 570843, \quad y = 234960, \quad z = 1128524;$$

and putting $b = 3$, $c = 1$ gives

$$x = 153076, \quad y = 570960, \quad z = 600357.$$

Neither of these solutions is in Lal and Blundon's [3] computer-generated list.

For $m_1 = \dfrac{(16 - A^2)^2}{16A^2}$ we have, if $D = 2$, that

$$\alpha = \frac{-509}{40}, \quad \beta = \frac{-1139}{78}, \quad w = \frac{-260681}{34320};$$

thus, $\quad a = -260681$, $b = 2$, $c = 1$, and $d = 34320$. This gives

$$\begin{aligned} x &= 3(295001)(226361) \\ y &= 6(260681)(34320) = 2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 29 \cdot 89 \cdot 101 \\ z &= 4(176041)(240479). \end{aligned}$$

We can also determine another set of solutions of 4.5 by writing

$$n^3 + n^2(3m_0 - 4) + n(3m_0^2 - 8m_0 + A^2) + t_0^2 = (Cn^2 + Bn + t_0)^2.$$

This gives

$$m = m_0\left(\frac{m_0^4 - 6A^2m_0^2 + 16A^2m_0 - 3A^4}{3m_0^4 - 16m_0^3 + 6A^2m_0^2 - A^4}\right)^2. \tag{4.9}$$

Equation (4.9) will again generate two infinite sets of two-parameter formulas.

5. It is clear that the sequences of parametric solutions given in this paper by (3.1), (4.7), and (4.9) rapidly lead to solutions of high degree with "large" values for x, y, and z. But we know from Lal and Blundon's list [3] that there are many smaller solutions, and so there must be other parametric solutions of smaller degree, like (2.5) and (2.6). Some other solutions of degree 8 or more are given in Kraitchik [2, Ch. 5]. For each such parametric solution x, y, and z, we have the derived solution given by $X = yz$, $Y = xz$, and $Z = xy$. This effectively doubles the number of formulas. Whether there are solutions of (2.3) which give these smaller solutions remains open. It seems intuitively clear that the number of parametric solutions of given degree is finite, but that this number increases with the degree. Unfortunately, we have no idea what this rate of increase might be.

6. Finally, we see, from (2.1), that

$$x^2 + y^2 + z^2 = c^4d^4D^2\left(\frac{D^2+1}{D}\right)^2\left(w^4 + \frac{8w^3}{\left(\frac{D^2+1}{D}\right)} + 2w^2 - \frac{8w}{\left(\frac{D^2+1}{D}\right)} + 1\right).$$

Therefore, putting $D + 1/D = A$ as before, we see that $x^2 + y^2 + z^2$ is square if

$$w^4 + \frac{8}{A}w^3 + 2w^2 - \frac{8}{A}w + 1 = \square.$$

This equation is similar to (2.3). If we change A into $4/A$, we can deduce rational solutions using (3.1), starting with $w_0 = 1/A$. Therefore, we can generate a sequence of two-parameter formulas for a cuboid with edges x, y, and z^2, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all square.

A perfect cuboid would exist if we could find rational w and $A = D + \frac{1}{D} \neq 2$, where D is also rational, such that

$$w^4 + 2Aw^3 + 2w^2 - 2Aw + 1 \quad \text{and} \quad w^4 + \frac{8}{A}w^3 + 2w^2 - \frac{8}{A}w + 1$$

are both square, or if we could determine a solution $w = w(A)$ satisfying both quartics. This, of course, seems unlikely, but the problem of perfect cuboids remains stubbornly open.

REFERENCES

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