ON PRIMITIVE PYTHAGOREAN TRIANGLES WITH EQUAL PERIMETERS

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Dedicated to my wife Pesia and my son John

1. Introduction

A triple (x, y, z) of natural numbers is called a *Pythagorean Triangle* if x, y, z satisfy the Pythagorean equation

$$x^2 + y^2 = z^2.$$

The triple (x, y, z) is a *Primitive Pythagorean Triangle* (PPT) if x, y, z have no common factor greater than 1. If x is assumed to be odd, the set of PPT's can be generated by the set of pairs of natural numbers (u, v) satisfying

$$u > v > 0, \quad (u, v) = 1, \quad u + v \equiv 1 \pmod 2, \tag{1}$$

the well-known generating formula being

$$(x, y, z) = (u^2 - v^2,\ 2uv,\ u^2 + v^2).$$

The pair (u, v) is called the *generator* of the PPT (x, y, z).

In terms of the generator, the *perimeter* S of (x, y, z), $S = x + y + z$, may be expressed as

$$S = 2u(u + v).$$

Denote by H the set of all such perimeters. Let H_k be the subset of H defined by the relation: $S \in H_k$ if S is the perimeter of *exactly* k PPT's.

It is not difficult to show that H_1 is an infinite set, i.e., there is an infinite set of PPT's each one of which has a perimeter not shared by any other PPT. The surprising fact that H_2 is also an infinite set is proved in [1]. It is the main purpose of this paper to prove that H_k is an infinite set for any k, $k \geq 3$; see Proposition 3.3 below. The proof may appear to be constructive, but it is ultimately seen to depend on a known existential Theorem of analytic number theory, the so-called modern version of Bertrand's postulate.

Necessary conditions for the construction of k PPT's with equal perimeters are given in the next section. That the conditions can be met is shown in the proof of Proposition 3.3.

2. A Constructive Device

Let us first construct k different generators (u, v) of PPT's with equal perimeters.

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Proposition 2.1: Let $B_1, B_2, \ldots, B_k$ be k ($k \geq 3$) odd positive integers, pairwise relatively prime, $B_1 < B_2 < \cdots < B_k$, and

$$B_k < B_1\sqrt{2}. \tag{2}$$

Let

$$A_k = \prod_{i=1}^{k} B_i \text{ and } u_t = A_k/B_t \text{ for } t \in T,\ T = \{1, 2, \ldots, k\}.$$

Assume there exists an odd positive integer P_k satisfying the two conditions

$$(P_k, u_t) = 1,\ t \in T, \tag{3}$$

$$\frac{B_2B_3 \ldots B_k}{B_1} < P_k < 2\,\frac{B_1B_2 \ldots B_{k-1}}{B_k}. \tag{4}$$

If $v_t = P_kB_t - u_t$, $t \in T$, then the pairs (u_t, v_t) are generators of k PPT's having equal perimeters S, $S = 2P_kA_k$.

Proof: We show first that (u_t, v_t) is the generator of a PPT for each $t \in T$, i.e., that (u_t, v_t) satisfies (1). From the definitions of u_t, v_t, it follows that

$$u_1 > u_2 > \cdots > u_k \quad \text{and} \quad v_1 < v_2 < \cdots < v_k. \tag{5}$$

Since by (4),

$$v_1 = P_kB_1 - u_1 = P_kB_1 - B_2B_3 \ldots B_k > 0,$$

it follows from (5) that $v_t > 0$ for $t \in T$. Moreover, it follows from (5) that $u_t > v_t$, $t \in T$, provided $u_k > v_k$. And this is a consequence of (4):

$$u_k - v_k = 2u_k - P_kB_k > (2A_k/B_k) - 2B_1B_2 \ldots B_{k-1} = 0.$$

Thus, $u_t > v_t$, $t \in T$.

Next, $(u_t, v_t) = 1$ if and only if $(u_t, u_t + v_t) = (A_k/B_t, P_kB_t) = 1$, which is true since, by assumption, $(u_t, P_k) = 1$ and the B_i's are pairwise relatively prime.

Since $u_t + v_t$ is odd, u_t and v_t must have opposite parity, i.e., $u_t + v_t \equiv 1 \pmod 2$. This concludes the proof that (u_t, v_t) satisfies (1) for each $t \in T$.

Finally, since $S = 2u_t(u_t + v_t) = 2P_kA_k$ is independent of t, the k PPT's generated by (u_t, v_t), $t \in T$, have equal perimeters.

3. Infinity of H_k

The main argument of this section rests on the following existential result; see [2], page 371.

Theorem 3.1: For every positive number ε there exists a number ξ such that for each x, $x > \xi$, there is a prime number between x and $(1 + \varepsilon)x$. (It will be used to prove the following proposition which has a certain interest in itself.)

Proposition 3.2: Let $k \geq 3$ and let $\delta > 0$. Then there is a number ξ such that for every y, $y > \xi$, there are k consecutive primes $B_1, B_2, \ldots, B_k$ and a prime P_k satisfying the inequalities

$$y < B_1 < B_2 < \cdots < B_k < \sqrt{1 + \delta}\, y,$$

$$\frac{B_2B_3 \ldots B_k}{B_1} < P_k < (1 + \delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k}.$$

Proof: Let ε_1 be a given number such that $0 < \varepsilon_1 < \sqrt{1 + \delta} - 1$. By Theorem 3.1, there is a number ξ_1 such that for every $x > \xi_1$, there are at least k consecutive primes $B_1, B_2, \ldots, B_k$ in the open interval $(x, (1 + \varepsilon_1)x)$. Let

$$\varepsilon = \frac{1 + \delta}{(1 + \varepsilon_1)^2} - 1$$

and take ξ_2 so large that for each x, $x > \xi_2$, there is at least one prime number in the interval $(x, (1 + \varepsilon)x)$.

Let $\xi = \max(\xi_1, \xi_2)$. Then for every y, $y > \xi$, we have that the interval $(y, (1 + \varepsilon_1)y)$ contains k consecutive primes,

$$y < B_1 < B_2 < \cdots < B_k < (1 + \varepsilon_1)y, \tag{6}$$

and the interval $(y, (1 + \varepsilon)y)$ contains a prime number $\overline{P}_k$,

$$y < \overline{P}_k < (1 + \varepsilon)y. \tag{7}$$

We show next that the interval

$$[X, Y] = \left[\frac{B_2B_3 \ldots B_k}{B_1}, (1 + \delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k}\right]$$

contains $\overline{P}_k$. On the one hand, we know from (7) that $[X, (1 + \varepsilon)X]$ contains at least the prime $\overline{P}_k$, since for $k \geq 3$, $X = B_2B_3 \ldots B_k/B_1 > B_2$ and $B_2 > y$ by (6). On the other hand, $[X, (1 + \varepsilon)X]$ is a subinterval of $[X, Y]$ if we show $(1 + \varepsilon)X < Y$. This last inequality is equivalent to

$$(1 + \delta)\frac{B_1B_2 \ldots B_{k-1}}{B_k} > \frac{1 + \delta}{(1 + \varepsilon_1)^2} \cdot \frac{B_2B_3 \ldots B_k}{B_1},$$

which, in turn, is equivalent to

$$(1 + \varepsilon_1)^2B_1^2 > B_k^2.$$

But $(1 + \varepsilon_1)B_1 > (1 + \varepsilon_1)y > B_k$ by (6). Thus $Y > (1 + \varepsilon)X$. This concludes the proof.

We are now ready to prove the main proposition.

Proposition 3.3: Let H_k, $k \geq 3$, be the set of integers S such that S is the perimeter of exactly k PPT's. Then H_k is infinite.

Proof: Taking $\delta = 1$ in Proposition 3.2, we can count on k consecutive primes $B_1, B_2, \ldots, B_k$ such that

$$B_k < \sqrt{2}B_1,$$

so condition (2) is satisfied; moreover there is a prime P_k such that condition (4) is satisfied.

Defining A_k, u_t, and v_t as in Proposition 2.1, we see that (3) is also satisfied, so we may conclude that (u_t, v_t), $t \in T$, generate k PPT's having equal perimeter $S = 2P_k A_k$.

Since y in (6) may be taken to be any number larger than ξ, it is clear that the above process may be repeated infinitely often. Each time we obtain a new set of k PPT's having equal perimeters.

It remains to show that no PPT, other than the ones constructed, can have perimeter $S = 2P_k A_k$. To do so, assume (u, v) generates a PPT with perimeter $S = 2P_k A_k$. We will show that (u, v) is not a generator of a PPT unless (u, v) is one of the pairs (u_t, v_t) constructed above.

Since $S = 2u(u + v) = 2P_k A_k = 2B_1 B_2 \ldots B_k P_k$, there are but a finite number of possible values for u and $u + v$. We assume first that P_k is a factor of u and consider the three possibilities:

(i) $u = P_k$, $u + v = B_1 B_2 \ldots B_k$,

(ii) $u = B_1 B_2 \ldots B_k P_k$, $u + v = 1$,

(iii) $u = q_1 q_2 \ldots q_m P_k$, $u + v = q_{m+1} q_{m+2} \ldots q_k$,

where $q_1 q_2 \ldots q_m$, $m \in \{1, 2, \ldots, k - 1\}$, denotes any one of the products of m different primes from the set $\{B_1, B_2, \ldots, B_k\}$, and $q_{m+1} q_{m+2} \ldots q_k$ the product of the remaining primes.

In case (i), condition (4) implies

$$2u = 2P_k < 4B_1 B_2 \ldots B_{k-1}/B_k < B_1 B_2 \ldots B_k = u + v,$$

so that $u < v$, a contradiction of (1).

For case (ii), $v = 1 - u < 0$, contradicting (1).

For case (iii), using (4), we write

$$\begin{aligned}(q_1 q_2 \ldots q_m)(q_{m+1} q_{m+2} \ldots q_k) P_k &= A_k P_k > A_k^2/B_1^2 \\ &= B_2^2 \ldots B_k^2 \geq (q_{m+1} q_{m+2} \ldots q_k)^2.\end{aligned}$$

Then

$$u = q_1 q_2 \ldots q_m P_k > q_{m+1} q_{m+2} \ldots q_k = u + v,$$

contradicting (1).

Next, we shall assume that P_k is not a factor of u. Then P_k must be a factor of $(u + v)$, and we consider the four possibilities:

(I) $u + v = P_k$, $u = B_1 B_2 \ldots B_k$,

(II) $u + v = B_1 B_2 \ldots B_k P_k$, $u = 1$,

(III) $u + v = q_{m+1} q_{m+2} \ldots q_k P_k$, $u = q_1 q_2 \ldots q_m$,

where $q_1 q_2 \ldots q_m$, $m \in \{1, 2, \ldots, k - 2\}$, denotes any one of the products of m different primes from the set $\{B_1, B_2, \ldots, B_k\}$, and $q_{m+1} q_{m+2} \ldots q_k$ the product of the remaining primes. Note that $u + v$ contains at least two of the primes B_i as factors.

(IV) $u + v = B_t P_k$, $u = B_1 B_2 \ldots B_{t-1} B_{t+1} \ldots B_k$, $t \in T$.

In case (I), using (4), we get

$$u + v = P_k < 2A_k/B_k^2 < B_1 B_2 \ldots B_k = u,$$

contradicting (1).

In case (II), $v = B_1B_2 \ldots B_kP_k - 1 > 1 = u$, contradicting (1).

For case (III), using (4), we have

$$u + v = q_{m+1}q_{m+2} \cdots q_kP_k > q_{m+1}q_{m+2} \cdots q_kA_k/B_1^2$$
$$= (q_{m+1}q_{m+2} \cdots q_k)^2(q_1q_2 \cdots q_m)/B_1^2 > 2q_1q_2 \cdots q_m = 2u,$$

a contradiction of (1).

Case (IV) is seen to describe the k pairs (u_t, v_t) defined above. These k pairs then generate k PPT's with equal perimeters $S = 2P_kA_k$, and no other PPT can have this perimeter.

4. Examples

Let us conclude with a few examples.

(1) When $k = 3$, we have:

B_1	B_2	B_3	P_3	(u_1, v_1)	(u_2, v_2)	(u_3, v_3)	S
11	13	15	19	(195, 14)	(165, 82)	(143, 142)	81,510
31	37	43	53	(1591, 52)	(1333, 628)	(1147, 1132)	5,228,026
17	19	21	25	(399, 26)	(357, 118)	(323, 202)	339,150
17	19	21	29	(399, 94)	(357, 194)	(323, 286)	393,414
23	25	29	33	(725, 34)	(667, 158)	(575, 382)	1,110,550
23	29	31	41	(899, 44)	(713, 476)	(667, 604)	1,695,514
23	29	31	43	(899, 90)	(713, 534)	(667, 666)	1,778,222
29	31	37	41	(1147, 42)	(1073, 198)	(899, 618)	2,727,566

(2) Finally, let $k = 4$ and

$B_1 = 17$, $B_2 = 19$, $B_3 = 21$, $B_4 = 23$.

For the integer P_4 within the bounds in (4), we can select any prime P_4 in the set

{541, 547, 557, 563, 569, 571, 577, 587};

moreover, Proposition 2.1 allows us to take any nonprime P_4 in the set

{545, 559, 565, 581, 583}.

References

1. Leon Bernstein. "Primitive Pythagorean Triples." *Fibonacci Quarterly* 20.3 (1982):227-241.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. Oxford: Oxford University Press, 1900.

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