CONGRUENCES FOR NUMBERS OF RAMANUJAN

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1. Introduction

In Chapter 3 of his second notebook [1, p. 164], Ramanujan defined polynomials

$$A_r(x) = \sum_{k=0}^{r-2} a(r, k)x^{2r-k-1} \quad (r \geq 2) \tag{1.1}$$

with $A_1(x) = x$. The numbers $a(r, k)$ are integers such that $a(2, 0) = 1$ and, for $r \geq 2$,

$$a(r + 1, k) = (r - 1)a(r, k - 1) + (2r - k - 1)a(r, k). \tag{1.2}$$

Also, $a(r, k) = 0$ when $k < 0$ or $k > r - 2$. Properties of $A_r(x)$, and the motivation for defining them, are discussed in [1, pp. 163-166]. Included in that reference is a list of the polynomials $A_r(x)$, $1 \leq r \leq 7$, and the following theorem:

$$\sum_{k=0}^{r-2} a(r, k) = A_r(1) = (r - 1)^{r-1}. \tag{1.3}$$

In [3] it was shown how $a(r, r - k)$ can be expressed in terms of Stirling numbers of the first kind, and the following special cases were worked out:

$$a(r, 0) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2r - 3), \tag{1.4}$$

$$a(r, 1) = [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2r - 3)](r - 2)/3, \tag{1.5}$$

$$a(r, r - 2) = (r - 2)! \tag{1.6}$$

We note here that it is easy to prove by induction that

$$a(r, 2) = (r - 3)(r - 2)(r - 1)5 \cdot 7 \cdot \cdots \cdot (2r - 5)/3.$$

The main purpose of the present paper is to prove congruences for $a(r, k)$ (mod p), where p is a prime number. As an application of some of these congruences we prove $A_p(x)/x^{p+1}$ and $A_{p-1}(x)/x^p$ are irreducible over the rational field. We also determine, for all r, the least residues of $a(r, k)$ (mod 2), (mod 3), and (mod 4). For each r we find the largest k such that $a(r, k) \not\equiv 0$ (mod p), and we make a conjecture, based on computer evidence, about the smallest k such that $a(r, k) \not\equiv 0$ (mod p). We also conjecture the following periodicity property:

$$a(r + (p - 1)p, k + (p - 2)p) \equiv a(r, k) \pmod{p}.$$

This has been verified for all primes $p \leq 251$. A few other results and conjectures are given for moduli not necessarily prime.

2. Congruences (Mod P)

Theorem 2.1: For any prime number p,

$$a(p, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 3),$$

$$a(p,\ p-2) \equiv 1 \pmod{p}.$$

Proof: In [1, p. 164] we have

$$A_r(x) = x(r-2)A_{r-1}(x) + x\sum_{k=1}^{r-1}\binom{r-1}{k}A_k(x)A_{r-k}(x), \tag{2.1}$$

and hence

$$A_{p+1}(x) \equiv -xA_p(x) + xA_p(x)A_1(x) \equiv (x^2 - x)A_p(x) \pmod{p}.$$

Comparing coefficients of x^{2p-k+1}, we have

$$a(p+1,\ k) \equiv a(p,\ k) - a(p,\ k-1) \pmod{p}. \tag{2.2}$$

From (1.2) we have

$$a(p+1,\ k) \equiv -a(p,\ k-1) - (k+1)a(p,\ k) \pmod{p}. \tag{2.3}$$

Combining (2.2) and (2.3), we see that

$$(k+2)a(p,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ \ldots,\ p-2). \tag{2.4}$$

The theorem now follows from (2.4) and (1.6). We note that results similar to Theorem 2.1 have been proved for the Stirling numbers [2, pp. 218-219].

Theorem 2.2: For any odd prime number p,

$$a(p-1,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ 1,\ \ldots,\ p-4),$$

$$a(p-1,\ p-3) \equiv (p-3)! \pmod{p}.$$

Proof: From (1.2) we have

$$a(p,\ k) \equiv -2a(p-1,\ k-1) - (k+3)a(p-1,\ k) \pmod{p}.$$

Thus, by Theorem 2.1,

$$(k+3)a(p-1,\ k) \equiv -2a(p-1,\ k-1) \pmod{p} \quad (k = 1,\ \ldots,\ p-3). \tag{2.5}$$

Since

$$a(p-1,\ 0) = 1 \cdot 3 \cdot \cdots \cdot (2p-5) \equiv 0 \pmod{p} \text{ for } p > 3,$$

the theorem follows from (2.5) and (1.6).

Theorem 2.3: For any odd prime number p, the polynomials

$$A_p(x)/x^{p+1} \quad \text{and} \quad A_{p-1}(x)/x^p$$

are irreducible over the rational field.

Proof: Assume $p > 2$. We know

$$a(p,\ k) \equiv 0 \pmod{p} \quad (k = 0,\ 1,\ \ldots,\ p-3),$$

$$a(p,\ 0) = 1 \cdot 3 \cdot \cdots \cdot (2p-3) \not\equiv 0 \pmod{p^2},$$

$$a(p,\ p-2) \equiv 1 \not\equiv 0 \pmod{p}.$$

Thus, $A_p(x)/x^{p+1}$ is irreducible by Eisenstein's Criteria. The proof is similar for $A_{p-1}(x)/x^p$.

We note here that Theorem 2.1 could be generalized by using p^j, $j \geq 1$, instead of p. Replacing p by p^j in the proof, we have

$$a(p^j,\ k) \equiv 0 \pmod{p} \quad (k \not\equiv -2 \pmod{p}).$$

Theorem 2.4: If p is an odd prime and if $m \geq p$, then

$$a(m, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 3).$$

If $m > p$, then

$$a(m, p - 2) = a(p + t, p - 2) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1) \pmod{p}.$$

Proof: We use induction on m. The first part of the theorem is true for $m = p$. Assume it is true for $m = p, p + 1, \ldots, r$. Then, by (1.2) we have, for $k = 0, 1, \ldots, p - 3$,

$$a(r + 1, k) \equiv 0 \pmod{p};$$

therefore, the first part of the theorem is true for all $m \geq p$. Now, by (1.2), if $t > 0$, then

$$\begin{aligned} a(p + t, p - 2) &\equiv (2t - 1)a(p + t - 1, p - 2) \\ &\equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1)a(p, p - 2) \\ &\equiv 1 \cdot 3 \cdot \cdots \cdot (2t - 1) \pmod{p}. \end{aligned}$$

This completes the proof.

We note that when $t \geq (p + 1)/2$,

$$a(p + t, p - 2) \equiv 0 \pmod{p} \quad (p > 2).$$

We also note the following special cases of Theorem 2.4. For $k = 0, 1, 2, \ldots, p - 3$:

$$\begin{aligned} &a(p + 1, k) \equiv 0 \pmod{p}; && a(p + 2, k) \equiv 0 \pmod{p}: \\ &a(p + 1, p - 2) \equiv 1 \pmod{p}; && a(p + 2, p - 2) \equiv 3 \pmod{p}; \\ &a(p + 1, p - 1) \equiv -1 \pmod{p}; && a(p + 2, p - 1) \equiv -2 \pmod{p}; \\ & && a(p + 2, p) \equiv 0 \pmod{p}. \end{aligned}$$

Theorem 2.5: Let p be an odd prime. Then, for $k = 0, 1, \ldots, 2p - 5$:

$$\begin{aligned} &a(2p, k) \equiv 0 \pmod{p}; && (2p - 1, k) \equiv 0 \pmod{p}; \\ &a(2p, 2p - 4) \equiv 1 \pmod{p}; && a(2p - 1, 2p - 4) \equiv 1 \pmod{p}; \\ &a(2p, 2p - 3) \equiv -2 \pmod{p}; && a(2p - 1, 2p - 3) \equiv 0 \pmod{p}; \\ &a(2p, 2p - 2) \equiv 0 \pmod{p}. && \end{aligned}$$

Proof: We know by (1.6) and Theorem 2.4 that

$$a(2p, 2p - 2) \equiv 0 \equiv a(2p, p - 2) \pmod{p}.$$

From (2.1) we have

$$A_{2p+1}(x) \equiv (-x + x^2)A_{2p}(x) + 2xA_p(x)A_{p+1}(x) \pmod{p}.$$

Thus, by Theorem 2.1 and Theorem 2.4 (with $m = p + 1$),

$$A_{2p+1}(x) \equiv (-x + x^2)A_{2p}(x) + 2x^{2p+5} - 2x^{2p+4} \pmod{p}. \tag{2.6}$$

Congruence (2.6) gives, for $k \neq 2p - 3, 2p - 4$,

$$a(2p + 1, k) \equiv a(2p, k) - a(2p, k - 1) \pmod{p}, \tag{2.7}$$

and from (1.2) we have

$$a(2p + 1, k) \equiv -(k + 1)a(2p, k) - a(2p, k - 1) \pmod{p}. \tag{2.8}$$

Combining (2.7) and (2.8), we have, for $k \neq 2p - 3, 2p - 4$,

$$(k + 2)a(2p, k) \equiv 0 \pmod{p}.$$

For $k = 2p - 3$ and $k = 2p - 4$, (2.6) and (2.8) give

$$(2p - 1)a(2p, 2p - 3) \equiv 2 \pmod{p},$$
$$(2p - 2)a(2p, 2p - 4) \equiv -2 \pmod{p},$$

and we see that the congruences for $a(2p, k)$ in Theorem 2.5 are valid. Now, by (1.2) and (1.4), we have

$$a(2p, k) \equiv -2a(2p - 1, k - 1) - (k + 3)a(2p - 1, k) \pmod{p},$$
$$a(2p - 1, 0) \equiv 0 \pmod{p}. \tag{2.9}$$

Thus, $a(2p - 1, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 4)$,

and by Theorem 2.4,

$$a(2p - 1, p - 3) \equiv 0 \pmod{p}.$$

It is now clear that the congruences for $a(2p - 1, k)$ follow from the congruences for $a(2p, k)$ and (2.9). This completes the proof.

Theorem 2.6: If p is prime and $m \geq 2p$, then

$$a(m, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 5),$$
$$a(m, 2p - 4) = a(2p + t, 2p - 4) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t + 1) \pmod{p}.$$

Proof: We use induction on m. The theorem is true for $m = 2p$. Assume it is true for $m = 2p, 2p + 1, \ldots, r$. Then, by (1.2), we have

$$a(r + 1, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 5);$$

therefore, the first part of the theorem is true for all $m \geq 2p$. By (1.2), we have, for $t > 0$,

$$\begin{aligned} a(2p + t, 2p - 4) &\equiv (2t + 1)a(2p + t - 1, 2p - 4) \\ &\equiv 3 \cdot 5 \cdot \cdots \cdot (2t + 1)a(2p, 2p - 4) \\ &\equiv 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2t + 1) \pmod{p}. \end{aligned}$$

This completes the proof.

Using the same sort of proof as the proof of the first part of Theorem 2.5, we can show, for $p > 2$,

$$a(3p, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 3p - 7;\ k \neq 2p - 2).$$

The case $a(3p, 2p - 2)$ has not been resolved. We indicate with Conjecture 1 in Section 4 what the general situation appears to be. The next theorem deals with a related problem, namely, the problem of finding the largest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Define $g(p, r)$ to be the largest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Theorem 2.7: Let r be a positive integer, $r \geq 2$. Write

$$r = 2 + (s(p - 1) + t)p + u$$

with $s \geq 0$, $0 \leq t \leq p - 2$, and $0 \leq u \leq p - 1$. Then

$$g(p, r) = m = sp(p - 2) + t(p - 1) + u.$$

Furthermore,

$$a(r, m) \equiv u!\ (p - 2)!/(p - 2 - t)! \pmod{p}.$$

Proof: We give a brief outline of the proof by induction on s, t, and u. Note that showing $a(r, k) \equiv 0 \pmod{p}$ for all $k > m$ is simple, and we omit the details. The recurrence relation (1.2) is the main tool in all of the following. The theorem is certainly true for $s = t = u = 0$. For fixed s and t, induction on u is straightforward. Note that the induction applies to arbitrarily large u; the statement of the theorem restricts u to the nonzero values of $a(r, m)$. If the theorem is true for some fixed value of s, $u = p - 1$, and some value of t, then it is not hard to show that the theorem must be true for the same s, $u = 0$, and the successor of t. By induction, if this theorem is true for some s and for $t = u = 0$, then it is also true for that s and all $0 \le t \le p - 2$ and $0 \le u \le p - 1$.

Now suppose the theorem is true for some s and all t and u such that $0 \le t \le p - 2$ and $0 \le u \le p - 1$. Let

$$r_0 = 2 + (s(p - 1) + (p - 2))p$$

and let

$$m_0 = sp(p - 2) + (p - 2)(p - 1).$$

Then, putting $t = p - 2$ in the induction hypothesis, we have

$$a(r_0 + u, m_0 + u) \equiv u! \pmod{p} \text{ for } 0 \le u \le p - 1.$$

Since $r_0 - 2 \equiv 0 \pmod{p}$ and $2r_0 - (m_0 + 1) - 3 \equiv 0 \pmod{p}$, we must have

$$a(r_0, m_0 - 1) \equiv 0 \equiv 0 \cdot 0! \pmod{p}.$$

Now induct on u to show that

$$a(r_0 + u, m_0 + u - 1) \equiv u \cdot u! \pmod{p}.$$

Finally, we can conclude that:

$$a(r_0 + p, m_0 + p) \equiv p! \equiv 0 \pmod{p};$$

$$a(r_0 + p, m_0 + p - 1) \equiv p \cdot p! \equiv 0 \pmod{p};$$

$$\begin{aligned} a(2 + (s + 1)(p - 1)p, (s + 1)p(p - 2)) &= a(r_0 + p, m_0 + p - 2) \\ &\equiv 0 \cdot a(r_0 + p - 1, m_0 + p - 3) + (4 - 0 - 3) \cdot a(r_0 + p - 1, m_0 + p - 2) \\ &\equiv (p - 1) \cdot (p - 1)! \equiv 1 \pmod{p}. \end{aligned}$$

It follows that the theorem is true for $t = u = 0$ and $s + 1$. By induction, the theorem is true for all $s \ge 0$, $0 \le t \le p - 2$, and $0 \le u \le p - 1$.

The proof of the following theorem follows the same lines as the proof of Theorem 2.2.

Theorem 2.8: Let p be prime, $p > 3$. Then, for $1 \le t \le (p - 3)/2$,

$$a(p - t, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 2t - 2).$$

For $2 \le t \le p - 1$,

$$a(2p - t, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, 2p - 2t - 2).$$

For example, using Theorem 2.8, Theorem 2.2, and (1.2), we have, for $p > 5$,

$$a(p - 2, k) \equiv 0 \pmod{p} \quad (k = 0, 1, \ldots, p - 6),$$

$$a(p - 2, p - 5) \equiv -(p - 4)!/3 \pmod{p},$$

$$a(p - 2, p - 4) \equiv (p - 4)! \pmod{p}.$$

3. Congruences (Mod 2), (Mod 3), and (Mod 4)

In this section we first determine when $a(r, k)$ is even and when it is odd.

Theorem 3.1:

$$a(r,\ 0) \equiv 1 \pmod 2 \quad (r \ge 2),$$
$$a(r,\ 1) \equiv r \pmod 2 \quad (r \ge 3),$$
$$a(r,\ k) \equiv 0 \pmod 2 \quad (k > 1).$$

Proof: The congruences for $a(r,\ 0)$ and $a(r,\ 1)$ are clear from (1.4) and (1.5). By (1.2) we have, for $k > 1$,

$$a(2r,\ k) \equiv (k + 1)a(2r - 1,\ k) \pmod 2.$$

If k is odd, we clearly have $a(2r,\ k)$ is even. If k is even, then

$$a(2r,\ k) \equiv a(2r - 1,\ k) \pmod 2.$$

And by (1.2), since $k - 1$ is odd,

$$a(2r - 1,\ k) \equiv a(2r - 2,\ k) \pmod 2.$$

Thus,

$$a(2r,\ k) \equiv a(2r - 2,\ k) \equiv \cdots \equiv a(k + 2,\ k) \equiv k! \equiv 0 \pmod 2.$$

Now since

$$a(2r + 1,\ k) \equiv a(2r,\ k - 1) + (k + 1)a(2r,\ k),$$

we have $a(2r + 1,\ k)$ is even if $k > 1$. This completes the proof.

The patterns (mod 4) and (mod 8) are suggested by the computer and can be proved by induction on r. For (mod 4) we have the following congruences.

Theorem 3.2: $a(r,\ k) \equiv 0 \pmod 4$ for all k except:

$$a(r,\ 0) \equiv \begin{cases} 1 \pmod 4 \text{ if } r \equiv 1 \text{ or } 2 & \pmod 4, \\ 3 \pmod 4 \text{ if } r \equiv 0 \text{ or } 3 & \pmod 4, \end{cases}$$

$$a(r,\ 1) \equiv \begin{cases} 1 \pmod 4 \text{ if } r \equiv 1 \text{ or } 3 & \pmod 4, \\ 2 \pmod 4 \text{ if } r \equiv 0 & \pmod 4, \end{cases}$$

$$a(r,\ 2) \equiv 2 \pmod 4 \text{ if } r \equiv 0 \pmod 4,$$
$$a(r,\ 3) \equiv 2 \pmod 4 \text{ if } r \equiv 1 \pmod 4.$$

Theorems 3.1 and 3.2 suggest the following, which can be proved by means of (1.2) and induction on n.

Theorem 3.3: If $k \ge 2n$, then $a(r,\ k) \equiv 0 \pmod{2^n}$.

To prove congruences (mod 3) we need the following lemma, which is a special case of Conjecture 4 of Section 4.

Lemma: For $r \ge 2$, $a(r,\ k) \equiv a(r + 6,\ k + 3) \pmod 3$.

Proof: The lemma is true for $r = 2$, since

$$a(8,\ 3) \equiv 1 \pmod 3,$$
$$a(8,\ k) \equiv 0 \pmod 3 \text{ if } k \ne 3.$$

Assume it is true for $r = m - 1$. Then, by (1.2),

$$a(m + 6,\ k + 3) \equiv (m - 2)a((m - 1) + 6,\ k + 2) + (2(m - 1) - 1 - k)a(m - 1,\ k)$$

(continued)

$$\equiv (m-2)a(m-1, k-1) + (2(m-1)-1-k)a(m-1, k)$$
$$\equiv a(m, k) \pmod 3.$$

Theorem 3.4: $a(r, k) \equiv 0 \pmod 3$ for all k except:

$$a\left(r, \left[\frac{r-1}{2}\right]\right) \equiv 1 \pmod 3, \quad r \ge 2,$$

$$a\left(r, \left[\frac{r+1}{2}\right]\right) \equiv \begin{matrix} r(r+1) \pmod 3 \text{ if } r \not\equiv 0 \pmod 6, \\ 1 \qquad \pmod 3 \text{ if } r \equiv 0 \pmod 6. \end{matrix}$$

Proof: Suppose $r \equiv 2 \pmod 6$, i.e., $r = 6j + 2$. Then, by the lemma,

$$a(r, k) \equiv a(6(j-1)+2, k-3) \equiv \cdots \equiv a(2, k-3j) \pmod 3.$$

Thus,

$$a(r, k) \equiv \begin{cases} 0 \pmod 3 \text{ if } k \ne 3j, \\ 1 \pmod 3 \text{ if } k = 3j = (r-2)/2. \end{cases}$$

The other cases of $r \pmod 6$ are handled in exactly the same way.

4. Conjectures

Theorem 2.4, Theorem 2.5, and information given by the computer suggest the following conjectures.

Conjecture 1: For all integers t and positive integers h such that $h + t \ge 1$,

$$a(hp + t, k) \equiv 0 \pmod p, \quad k = 0, 1, \ldots, h(p-2) - 1,$$

$$a(hp + t, h(p-2)) \equiv 1 \cdot 3 \cdot \cdots \cdot (2t + 2h - 3) \pmod p.$$

For $t \ge 0$, Conjecture 1 has already been proved in Section 2 of this paper for $h = 1$, $h = 2$. If we try induction and assume true for $h = m - 1$, we can show, as in Theorem 2.1 and Theorem 2.5,

$$(k+2)a(mp, k) \equiv 0 \pmod p \quad (k = 0, \ldots, m(p-2) - 2).$$

Thus, the proof depends on showing

$$a(mp, k) \equiv 0 \pmod p \text{ if } k \equiv -2 \pmod p.$$

The rest of the proof, for $t > 0$, would then follow. The cases $t \ge 0$ have been verified by the computer for all primes less than or equal to 251. The case $h + t = 0$ leads to the next conjecture.

Conjecture 2: Let p be any prime.

(i) Let h be any nonnegative integer. Then

$$a(2 + hp(p-1), m) \equiv 0 \pmod p \text{ if } m \ne hp(p-2).$$

(ii) Let h be a nonnegative integer, $h \not\equiv 0 \pmod p$. Then

$$a(1 + h(p-1), m) \equiv 0 \pmod p \text{ if } m \ne h(p-2).$$

(iii) Let h be a nonnegative integer, $h \not\equiv 0$ or $p - 1 \pmod p$. Then

$$a(h(p-1), m) \equiv 0 \pmod p \text{ if } m \ne h(p-2) - 1.$$

By Theorem 2.7, we know:

(i) $a(2 + hp(p-1), hp(p-2)) \equiv 1 \pmod p$.

(ii) Let $h = sp + t$, $1 \le t \le p - 1$, $s \ge 0$, then

$$\begin{aligned} &a(1 + h(p - 1), h(p - 2)) \\ &= a(2 + (s(p - 1) + (t - 1))p + (p - 1) - t), \\ &\quad sp(p - 2) + (t - 1)(p - 1) + (p - 1) - t) \\ &\equiv (p - 1 - t)! \cdot (p - 2)!/(p - 2 - (t - 1))! \equiv 1 \pmod{p}. \end{aligned}$$

(iii) Let $h = sp + t$, $1 \le t \le p - 2$, $s \le 0$, then

$$\begin{aligned} &a(h(p - 1), h(p - 2) - 1) \\ &= a(2 + (s(p - 1) + (t - 1))p + (p - 2) - t), \\ &\quad sp(p - 2) + (t - 1)(p - 1) + (p - 2) - t) \\ &\equiv (p - 2 - t)! \cdot (p - 2)!/(p - 2 - (t - 1))! \equiv (p - 2)!/(p - 1 - t). \end{aligned}$$

The authors are grateful to the referee for suggesting the next conjecture. Part of this conjecture would follow from Conjectures 1 and 2.

Define $f(p, r)$ to be the smallest k such that $a(r, k) \not\equiv 0 \pmod{p}$.

Conjecture 3: Clearly $f(p, r) = 0$ if $r \le (p + 1)/2$. Thus, for $r \ge 2$:

(i) $f(p, r) = (p - 2)\left[\dfrac{r}{p - 1}\right] - 1$ if $(p - 1) \mid r$;

(ii) $f(p, r) = (p - 2)\left[\dfrac{r}{p - 1}\right]$ if $r \equiv t \pmod{p - 1}$, $1 \le t \le (p + 1)/2$;

(iii) $f(p, r) \ge (p - 2)\left[\dfrac{r}{p - 1}\right] + 2t$ if $r \equiv t + (p + 1)/2 \pmod{p - 1}$, $1 \le t \le (p - 5)/2$.

In some cases, $f(p, r)$ is larger than the formula given in (iii) above. For example, $f(11, 17) = 13$, $f(11, 48) = 42$, $f(13, 22) = 19$, and $f(13, 68) = 59$ are larger by 2, and $f(41, 350) = 334$, $f(43, 1743) = 1703$, $f(61, 2152) = 2111$, and $f(67, 2038) = 2002$ are larger by 4. It appears to be difficult to predict when $f(p, r)$ will be larger than the formula or by how much it will be bigger. There are many cases where $f(p, r)$ is larger by 2 or 4, and we suspect the formula could be off by even more for very large primes.

Conjecture 4: If p is any prime, then

$$a(r + p(p - 1), m + p(p - 2)) \equiv a(r, m) \pmod{p} \text{ for any } r \ge 2,\ m \ge 0.$$

Because of the recursion formula, (1.2), it suffices to show that

$$a(2 + p(p - 1), m + p(p - 2)) \equiv a(2, m) \pmod{p}$$

for all integers m. In this manner, Conjecture 4 has been proved on the computer for any prime $p \le 251$.

Conjecture 5: If p is any prime and $m \ge 0$, then

$$a(r + p^n(p - 1), m + p^n(p - 2)) \equiv a(r, m) \pmod{p^n}$$

for all sufficiently large r.

Conjecture 5 has been proved on the computer for p^n up to 2^{13}, 3^7, 5^5, 7^4, 11^3, 13^3, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2.

Conjecture 6: If p is an odd prime, then

(i) $a(r, 0) \equiv p \pmod{2p}$ for $r \ge (p + 3)/2$;

(ii) $a(r, 1) \equiv \begin{cases} 0 \pmod{2p} & \text{if } r \text{ is even} \\ p \pmod{2p} & \text{if } r \text{ is odd} \end{cases}$ and $r \geq (p + 3)/2$;

(iii) if $m \geq 2$ and $r \geq (p + 3)/2$, then

$$a(r + p(p - 1), m + p(p - 2)) \equiv a(r, m) \pmod{2p}.$$

Conjecture 6 can be proved to be true if Conjecture 4 is assumed to be true. Similar conjectures for other composite moduli also seem to hold, but are more complicated to state.

5. Concluding Remarks

Apparently not much is known about the numbers $a(r, k)$. It would be useful if a generating function and a combinatorial interpretation were found. Also, it appears difficult to find values of $A_r(x)$ for $x \neq 0$, $x \neq 1$. We remark that it is easy to find derivative formulas for $A_r(x)$, however. It follows from (1.2) and the definition of $A_r(x)$ that

$$x^3 A_r'(x) = A_{r+1}(x) - (r - 1)xA_r(x),$$

and thus it is easy to find a general formula for $A_r^{(j)}(x)$. For example, we have by (1.3) and the above comments,

$$A_r'(1) = r^r - (r - 1)^r$$

$$A_r''(1) = (r + 1)[(r + 1)^r - 2r^r + (r - 1)^r].$$

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