FIBONACCI-LIKE MATRICES

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(Submitted February 1987)

It is well known that the powers of the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are matrices of the form

$$\begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$

whose entries are the Fibonacci numbers. If these matrices are normalized by dividing by F , so that the first entry is always 1, then the resulting sequence converges to

$$\begin{bmatrix} 1 & f \\ f & f^2 \end{bmatrix} \tag{1}$$

where $f = (1 + \sqrt{5})/2$ is the golden ratio.

Moore [2] noticed that if the 2 in M is replaced by $1 + x$ and the same procedure (taking powers of M and normalizing to obtain a 1 in the first entry) is performed then the resulting sequence seems to converge to a matrix of the same form with $f = (x + \sqrt{x^2 + 4})/2$. These observations naturally suggest the following questions.

1. If we start with any symmetric 2×2 matrix M, with positive integral entries, does a similar phenomenon occur and, if so, what is the corresponding value of f?

2. If there is convergence with $f = (a + \sqrt{d})/b$, what are the values of d that can occur, i.e., in what quadratic number fields do we find such f?

Since we normalize at each step, we can assume that

$$M = \begin{bmatrix} 1 & y \\ y & 1 + x \end{bmatrix}.$$

Diagonalizing M gives $M = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & 1 \\ (x + \sqrt{d})/2y & (x - \sqrt{d})/2y \end{bmatrix}, \quad D = \begin{bmatrix} (x + 2 + \sqrt{d})/2 & 0 \\ 0 & (x + 2 - \sqrt{d})/2 \end{bmatrix},$$

where $d = x^2 + 4y^2$. Thus we have $M^n = PD^nP^{-1}$. When we normalize D to make the leading entry 1, the second diagonal entry is less than one and so the sequence of its powers converges to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and so $PD\,P^{-1}$, when normalized, can be seen to converge to a matrix of the form (1) with $f = (x + \sqrt{d})/2y$, where $d = x^2 + 4y^2$.

The set of d that can be written as $x^2 + 4y^2$ can be easily found, since we know what numbers can be written as the sum of two squares. (A positive integer is the sum of two squares if, when factored, all its prime factors congruent to 3 modulo 4 occur with even exponent, see, e.g., [1].) If d is odd, then d is the sum of two squares if and only if it is of the form $x^2 + (2y)^2 = x^2 + 4y^2$, since one of the terms must be even. If d is even, then $d = x^2 + 4y^2$ if and only if $d = 4m$, where m can be written as the sum of two squares. Since d is even, x is even, and so d is divisible by 4. The 4 can then be factored out giving m as the sum of two squares. The converse is also easy. Thus, d is of the form $x^2 + 4y^2$ exactly if, when factored, all its odd prime factors congruent to 3 modulo 4 occur with even exponent and 2 does not occur with exponent 1.

Acknowledgment

The author would like to thank the editor for pointing out references [3], [4], and [5] which are also concerned with Moore's conjecture.

References

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